Some remarks on adiabatic time evolution and quasi-static processes in translation-invariant quantum systems

Vojkan Jaksic
McGill University

Based on joint work Claude-Alain Pillet and Clément Tauber

December 12, 2022

ADIABATIC THEOREM IN QUANTUM MECHANICS

- Born and Fock (1928), Kato (1958)

- Setting: $\mathcal{H}$, H, C^2 self-adjoint map

$$[0,1] \ni \tau \mapsto V(\tau) \in \mathcal{B}(\mathcal{H})$$

- Time-dependent Hamiltonian

$$H(\tau) = H + V(\tau).$$

- $T > 0$, the non-autonomous Schrödinger equation

$$\mathrm{i}\partial_t \psi(t) = (H + V(t/T))\psi(t),$$

$$\psi(s) = f \in \mathrm{Dom}(H),\ s,t \in [0,T].$$

- Solution: $\psi(t) = U_T^{s/T \to t/T} f$, U_T unitary propagator associated to

$$[0,1] \ni \tau \mapsto TH(\tau)$$

- Gap condition: For $\tau \in [0,1]$, $E(\tau)$ is an eigenvalue of finite multiplicity of $H(\tau) = H + V(\tau)$ such that

$$\inf_{\tau \in [0,1]} \mathrm{dist}(E(\tau), \mathrm{sp}(H(\tau)) \setminus \{E(\tau)\}) > 0$$

$P(\tau)$ associated projection.

KATO'S THEOREM

As $T \uparrow \infty$,

$$\sup_{\tau \in [0,1]} \|(I - P(\tau))U_T^{0 \to \tau} P(0)\| = O(T^{-1}).$$

Many refinements (under gap condition): Avron-Seiler-Yaffe, Nenciu, Joye-Pfister, Joye, J-Segert ...

The ultimate refinement dispenses with gap assumption: Avron-Elgart (1999), with an important technical comment by Teufel (2001).

AVRON-ELGART ADIABATIC THEOREM WITHOUT A GAP

Assumption: There exists a C^2 map

$$[0,1] \ni \tau \mapsto P(\tau) \in \mathcal{B}(\mathcal{H})$$

such that, for Lebesgue a.e. $\tau \in [0,1]$, $P(\tau)$ is the orthogonal projection onto the eigenspace of a finite multiplicity eigenvalue of $H(\tau)$.

Theorem: Then

$$\lim_{T\to\infty} \sup_{\tau\in[0,1]} \|(I - P(\tau))U_T^{0\to\tau} P(0)\| = 0.$$

ADIABATIC THEOREM IN QUANTUM STATISTICAL MECHANICS

- $(\mathcal{O}, \alpha)$ C^*-dynamical system, $\alpha^t = \mathrm{e}^{t\delta}$.

- ω is (α, β)-KMS state at inverse temperature $\beta > 0$ if

$$\omega(AB) = \omega(B\alpha^{\mathrm{i}\beta}(A)).$$

- $V = V^* \in \mathcal{O}$. Perturbed dynamics $\alpha_V^t = \mathrm{e}^{t\delta_V}$, $\delta_V = \delta + \mathrm{i}[V, \cdot]$.

- ω_V the perturbed (α_V, β)-KMS state.

The quantum dynamical system $(\mathcal{O}, \tau_V, \omega_V)$ is called ergodic/has the property of return to equilibrium if

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \omega_V(B^* \tau_V^t(A) B) \mathrm{d}t = \omega_V(B^* B) \omega_V(A)$$

for all $A, B \in \mathcal{O}$.

Robinson (1973).

Renewal of interest 1996– with focus on Pauli-Fierz systems (finite quantum system coupled to a free gas thermal reservoir)

NON-AUTONOMOUS SETTING

- Time-dependent perturbation: C^2 self-adjoint map

$$[0,1] \ni \tau \mapsto V(\tau) \in \mathcal{O}$$

- For $T > 0$, we consider the Cauchy problem for the non-autonomous Heisenberg equation

$$\partial_t \gamma^t(A) = \gamma^t \circ \delta_{V(t/T)}(A),$$

$\gamma^s(A) = A \in \mathrm{Dom}\,(\delta)$, $s, t \in [0, T]$.

$$\delta_{V(t/T)} = \delta + \mathrm{i}[V(t/T), \cdot].$$

- Solution: $\gamma^t = \alpha_T^{s/T \to t/T}$, where

$$[0,1] \times [0,1] \ni (\sigma, \tau) \mapsto \alpha_T^{\sigma \to \tau}$$

is the propagator generated by the time-dependent derivation

$$[0,1] \ni \tau \mapsto T(\delta + \mathrm{i}[V(\tau), \cdot])$$

$$\alpha_T^{\sigma \to \tau}(A) = \sum_{n=0}^{\infty} T^n \int_{\sigma \le \sigma_1 \le \cdots \le \sigma_n \le \tau} \mathrm{i}[\alpha^{(\sigma_1 - \sigma)T}(V(\sigma_1)),$$

$$\mathrm{i}[\cdots, \mathrm{i}[\alpha^{(\sigma_n - \sigma)T}(V(\sigma_n)), \alpha^{(\tau - \sigma)T}(A)] \cdots]]\mathrm{d}\sigma_1 \cdots \mathrm{d}\sigma_n.$$

ADIABATIC THEOREM IN QSM (FOR LOCAL PERURBATIONS)

Theorem: If, for Lebesgue a.e. $\tau \in [0,1]$, the quantum dynamical system $(\mathcal{O}, \alpha_{V(\tau)}, \omega_{V(\tau)})$ has the property of return to equilibrium, then

$$\lim_{T\to\infty} \sup_{\tau\in[0,1]} \|\omega_{V(0)} \circ \alpha_T^{0\to\tau} - \omega_{V(\tau)}\| = 0.$$

The proof (Fröhlich and Abou-Salem, J-Pillet) is a simple combination of the Avron–Elgart gapless adiabatic theorem and Araki's perturbation theory of the modular structure.

The last theorem has been a starting starting point of other developments:
Landauer principle and its refinements
Crooks fluctuation relation
Characterization of the KMS condition in terms of adiabaticity (Narnhofer-Thirring (1982)...)
.... Forthcoming review by Benoist-Fraas-J-Pillet (2023).

The entropic extension of QSM adiabatic theorem play particularly important role. The relative entropy of two density matrices is

$$S(\rho|\nu) = \mathrm{tr}(\rho(\log\rho - \log\nu)) \geq \frac{1}{2}\|\rho - \nu\|.$$

Extension to the general setting of QSM goes back to Araki (1976,1978).

Theorem. The following statement statements are equivalent:

1.

$$\lim_{T\to\infty} \sup_{\tau\in[0,1]} S\left(\omega_{V(0)} \circ \alpha_T^{0\to\tau} \middle| \omega_{V(\tau)}\right) = 0.$$

2.

$$\lim_{T\to\infty} \sup_{\tau\in[0,1]} \|\omega_{V(0)} \circ \alpha_T^{0\to\tau} - \omega_{V(\tau)}\| = 0.$$

Theorem. The following statements are equivalent:

1. For Lebesgue a.e. $\tau \in [0,1]$,

$$\lim_{T\to\infty} \omega_{V(0)} \circ \alpha_T^{0\to\tau}(\partial_\tau V(\tau)) = \omega_{V(\tau)}(\partial_\tau V(\tau)).$$

2. For all $\tau \in [0,1]$,

$$\lim_{T\to\infty} S\left(\omega_{V(0)} \circ \alpha_T^{0\to\tau} \middle| \omega_{V(\tau)}\right) = 0.$$

3. For all $\tau \in [0,1]$,

$$\lim_{T\to\infty} \|\omega_{V(0)} \circ \alpha_T^{0\to\tau} - \omega_{V(\tau)}\| = 0.$$

Proofs. The entropy balance equation

$$S\left(\omega_{V(0)} \circ \alpha_T^{0\to\tau} \middle| \omega_{V(\tau)}\right)$$
$$= \beta \int_0^\tau \left(\omega_{V(0)} \circ \alpha_T^{0\to\sigma} - \omega_{V(\sigma)}\right)(\partial_\sigma V(\sigma))\mathrm{d}\sigma$$

QUANTUM SPIN SYSTEMS

- $\mathcal{H}_x = \mathbb{C}^M$ single spin Hilbert space, $x \in \mathbb{Z}^d$.

- $\mathcal{F}$–the collection of all finite subsets of $\mathbb{Z}^d$.
  For $X \in \mathcal{F}$,

$$\mathcal{H}_X = \otimes_{x \in X} \mathcal{H}_x, \qquad \mathfrak{U}_X = \mathcal{B}(\mathcal{H}_X)$$

  $\mathfrak{U} \equiv$ the inductive limit C^*-algebra over the family $\{\mathfrak{U}_X\}_{X \in \mathcal{F}}$.

- $\mathcal{S}$–the set of states on $\mathfrak{U}$.
  $\mathcal{S}_I$– the set of translationally invariant states.

- Λ denotes the box centred at the origin.

- ρ_Λ – the restriction of ρ to $\mathfrak{U}_\Lambda$.The specific entropy of $\rho \in \mathcal{S}_I$:

$$s(\rho) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{S(\rho_\Lambda)}{|\Lambda|}$$

$S(\rho_\Lambda) = -\mathrm{Tr}\,(\rho_\Lambda \log \rho_\Lambda)$.

$s(\rho) \in [0, \log M]$.

- The specific relative entropy of $\rho, \omega \in \mathcal{S}_I$ is

$$s(\rho|\omega) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{S(\rho_\Lambda|\omega_\Lambda)}{|\Lambda|}$$

$S(\rho_\Lambda|\omega_\Lambda) = \mathrm{Tr}\,(\rho_\Lambda(\log \rho_\Lambda - \log \omega_\Lambda))$.

- Translationally invariant interaction $\{\Phi(X)\}_{X\in\mathcal{F}}$, $\Phi(X)=\Phi^*(X)\in\mathfrak{U}_X$, satisfying

$$\|\Phi\|_r=\sum_{0\in X}\mathrm{e}^{r(|X|-1)}\|\Phi(X)\|<\infty$$

The set $\mathcal{B}^r$ of such interactions is a Banach space.

From now on $\Phi\in\mathcal{B}^r$.

- Local Hamiltonians

$$H_\Lambda(\Phi)=\sum_{X\subset\Lambda}\Phi(X)$$

- Dynamics:

$$\tau_\Phi^t(A)=\lim_{\Lambda\uparrow\mathbb{Z}^d}\mathrm{e}^{\mathrm{i}tH_\Lambda(\Phi)}A\mathrm{e}^{-\mathrm{i}tH_\Lambda(\Phi)}$$

SPECIFIC ENERGY OBSERVABLE

$$E_\Phi = \sum_{X \ni 0} \frac{\Phi(X)}{|X|} \in \mathfrak{U}$$

One has

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \omega\left(H_\Lambda(\Phi)\right) = \omega(E_\Phi),$$

for any $\omega \in \mathcal{S}_I$. E_Φ is the **specific energy** observable of the interaction Φ.

TIME-DEPENDENT QUANTUM SPIN SYSTEMS

We now consider C^1 time-dependent interaction

$$\Psi : [0,1] \ni \tau \mapsto \Psi_\tau \in \mathcal{B}^r.$$

Local time-dependent Hamiltonians

$$H_\Lambda(\Psi_\tau) = \sum_{X \subseteq \Lambda} \Psi_\tau(X)$$

generate a non-autonomous C^*-dynamics $\alpha_{\Psi,\Lambda}^{\sigma \to \tau}$ on $\mathfrak{U}$.

Thermodynamic limit $\Rightarrow$ propagator $\alpha_\Psi^{\sigma \to \tau}$.

Entropy conservation (Lanford-Robinson): For any $\nu \in \mathcal{S}_I(\mathfrak{U})$ and $\sigma, \tau \in [0,1]$,

$$s(\nu) = s(\nu \circ \alpha_\Psi^{\sigma \to \tau}).$$

We consider a map

$$[0,1] \ni \tau \mapsto (\nu_\tau, \Psi_\tau) \in \mathcal{S}_I(\mathfrak{U}) \times \mathcal{B}^r$$

such that ν_τ is β-KMS for the "frozen" α^τ_Ψ, $\tau \in [0,1]$. We assume $\beta \ll 1$.

We further consider the non-autonomous time evolution on $\mathfrak{U}$ defined by the Cauchy problem

$$\partial_t \gamma^t(A) = \gamma^t \circ \delta_{\Psi_{t/T}}(A), \qquad \gamma^s(A) = A \in \mathfrak{U}_{\mathrm{loc}}$$

$s, t \in [0, T]$ in the adiabatic limit $T \to \infty$.

A rescaling of the time variables s and t gives

$$\gamma^t = \alpha^{s/T \to t/T}_{T\Psi},$$

where $\{\alpha^{\sigma \to \tau}_{T\Psi}\}_{\sigma,\tau \in [0,1]}$ is the propagator generated by $T\Psi$.

THEOREM

The following statements are equivalent:

1. For Lebesgue a.e. $\tau \in [0,1]$,

$$\lim_{T\to\infty} \nu_0 \circ \alpha_{T\Psi}^{0\to\tau}(\partial_\tau E_{\Psi_\tau}) = \nu_\tau(\partial_\tau E_{\Psi_\tau}).$$

2. For all $\tau \in [0,1]$,

$$\lim_{T\to\infty} s(\nu_0 \circ \alpha_{T\Psi}^{0\to\tau} | \nu_\tau) = 0.$$

3. For all $\tau \in [0,1]$,

$$\lim_{T\to\infty} \nu_0 \circ \alpha_{T\Psi}^{0\to\tau} = \nu_\tau.$$

Moreover, any of the above statements implies that

$$s(\nu_\tau) = s(\nu_0)$$

for all $\tau \in [0, 1]$.

Severe constraint on possibly validity of adiabatic theorem in the quantum spin system/fermion setting.

No go observation: Adiabaticity and approach to equilibrium are incompatible (talk at Polytechnico, June 2022).

TOPICS FOR DISCUSSION

- Status of adiabaticity in the translation invariant lattice setting of fermionic/quantum spin systems,

- At zero temperature, with gap assumption: powerful result of Bachmann-De Roeck-Fraas.

- At positive temperature, the status us unclear, and the presented results suggest that novel frameworks/interpretations might be needed.

- Comparison with the recent preprint Greenblatt-Lange-Marcelli-Porta ”Adiabatic Evolution of Low-Temperature Many-Body Systems”.