Norm approximations for the Fröhlich dynamics

Nikolai Leopold
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based on
[arXiv:2207.01598]
and
[Pure Appl. Anal. 3(4), 653–676 (2021)][1].

Itinerant Quantum Math Meetings
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[1] joint work with D. Mitrouskas, S. Rademacher, B. Schlein and R. Seiringer.

Outline of the talk

- Fröhlich model
- Landau-Pekar equations
- Mean-field regime
- Sketch of the proof
- Strong coupling regime

The Fröhlich Model

$$i\partial_t \Psi_{N,t} = H_N^{\mathrm{F}} \Psi_{N,t},$$

$$H_N^{\mathrm{F}} = \sum_{j=1}^{N} \left[-\Delta_j + \sqrt{\alpha} \int d^3k \, \left(G_{x_j}(k) a_k^* + \overline{G_{x_j}(k)} a_k \right) \right] + \underbrace{\int d^3k \, a_k^* a_k}_{=\mathcal{N}_a},$$

$$[a_k, a_l^*] = \delta(k-l) \quad \text{and} \quad [a_k, a_l] = [a_k^*, a_l^*] = 0.$$

Remarks:

- $\mathcal{H}^{(N)} = L^2(\mathbb{R}^{3N}) \otimes \mathcal{F}$ with $\mathcal{F} = \bigoplus_{n \geq 0} L^2(\mathbb{R}^3)^{\otimes_s^n}$,
- $G_x(k) = |k|^{-1} e^{-2\pi i k \cdot x} \notin L^2(\mathbb{R}^3)$ but $e^{-iH_N^{\mathrm{F}} t}$ can be defined via the associated quadratic form,

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Coherent states

Weyl operator: $W(\varphi) = e^{a^*(\varphi) - a(\varphi)}$ with $a(\varphi) = \int d^3k\, \overline{\varphi(k)} a_k$.
Coherent states: $\Psi_{N,t} = \Upsilon_{N,t} \otimes W(\sqrt{N}\varphi_t)\Omega$.
Facts:

- coherent states are eigenstates of the annihilation operator,
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a_k \Psi_{N,t} &= e^{-iH_N^{\mathrm{F}} t} a_k \Psi_{N,0} \\
&\quad - i \int_0^t ds \, e^{-iH_N^{\mathrm{F}}(t-s)} \left[a_k + |k|^{-1} N^{-1/2} \sum_{j=1}^{N} e^{-2\pi i k \hat{x}_j} \right] \Psi_{N,s}
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&\approx \sqrt{N} \left[\varphi_0(k) - i \int_0^t ds \left(\varphi_s(k) + |k|^{-1} \left\langle \psi_s, e^{-2\pi i k \cdot} \psi_s \right\rangle \right) \right] \Psi_{N,t}.
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The Landau-Pekar equations

Let $(\psi_t, \varphi_t) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ satisfy

$$\begin{cases} i\partial_t \psi_t(x) & = [-\Delta + \Phi_{\varphi_t}(x)]\, \psi_t(x) \\ i\partial_t \varphi_t(k) & = \varphi_t(k) + |k|^{-1} \int d^3x\, e^{-i2\pi k\cdot x}\, |\psi_t(x)|^2 \end{cases} \tag{LP}$$

with $(\psi, \varphi) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. Here, $h_{\varphi_t} = -\Delta + \Phi_{\varphi_t} - \langle \psi_t, \Phi_{\varphi_t} \psi_t \rangle$ with

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- interaction between the condensate wave function and a classical phonon field,
- nonlinear equations,
- well-posedness was shown by [Frank, Gang '15].

Result for reduced densities

One particle reduced density matrix: $\gamma^{(1,0)}_{\Psi_{N,t}} = \mathrm{Tr}_{2,\dots,N} \otimes \mathrm{Tr}_{\mathcal{F}} \left|\Psi_{N,t}\right\rangle \left\langle \Psi_{N,t}\right|$.

[L.'22]: Let (ψ_t, φ_t) be the solution of (LP) with i.d. $(\psi, \varphi) \in H^3(\mathbb{R}^3) \times L^2(\mathbb{R}^3, (1+|k|^2)^2 dk)$ s.t. $\|\psi\|_2 = 1$. Let $\Psi_N \in \mathcal{D}\big((H_N^0)^{1/2}\big)$ s.t. $\|\Psi_N\| = 1$ and $\Psi_{N,t} = e^{-iH_N^{\mathrm{F}} t}\Psi_N$. Then, there exists a constant $C > 0$ s.t

$$N^{-1}\big\langle W^*(\sqrt{N}\varphi_t)\Psi_{N,t}, \mathcal{N}_a\, W^*(\sqrt{N}\varphi_t)\Psi_{N,t}\big\rangle \leq \big(a[\Psi_N, \psi, \varphi] + 1/N\big)\, C e^{C|t|^9},$$

$$\mathrm{Tr}_{L^2(\mathbb{R}^3)} \left|\sqrt{1-\Delta}\big(\gamma^{(1,0)}_{\Psi_{N,t}} - |\psi_t\rangle\langle\psi_t|\big)\sqrt{1-\Delta}\right| \leq \big(a[\Psi_N, \psi, \varphi] + 1/N\big)^{1/2} C e^{C|t|^9}.$$

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- [L., Mitrouskas, Seiringer '21]: previous results for $\mathcal{D}(H_N^0)$ and $\mathcal{D}(H_N^F)$. Note that $\mathcal{D}(H_N^F) \cap \mathcal{D}(H_N^0) = \{0\}$ (see [Griesemer, Wünsch '16]).

Quantum fluctuations

Decomposition: $\Psi_{N,t} = W(\sqrt{N}\varphi_t) \sum_{k=0}^{N} \psi_t^{\otimes(N-k)} \otimes_s \chi_{\leq N}^{(k)}(t)$.

Unitary mapping: $U_N(t) : \mathcal{H}^{(N)} \to \left(\bigoplus_{k=0}^{N} L^2_{\perp\psi_t}(\mathbb{R}^3)^{\otimes_s k}\right) \otimes \mathcal{F}, \Psi_{N,t} \mapsto \left(\chi_{\leq N}^{(k)}(t)\right)_{k=0}^{N}$.

Schrödinger equation: $i\partial_t \chi_{\leq N}(t) = H(t)\chi_{\leq N}(t)$ with $H(t)$ being defined on $\mathcal{F} \otimes \mathcal{F}$ as

$$\begin{aligned} H(t) = &\int d^3x \, b_x^* h_{\varphi_t} b_x + \mathcal{N}_a \\ &+ \int d^3x \int d^3k \, K(t,k,x) \, (a_k^* + a_{-k}) \, b_x^* \left[1 - N^{-1}\mathcal{N}_b\right]_+^{1/2} + \text{h.c.} \\ &+ N^{-1/2} \int d^3x \, b_x^* \left(q(t)\hat{\Phi}q(t) - \langle \psi_t, \hat{\Phi}\psi_t \rangle\right) b_x . \end{aligned}$$

Here, $K(t,k,x) = \int d^3y \, q(t,x,y) \, |k|^{-1} \, e^{-2\pi iky} \psi_t(y)$ with $q(t) = 1 - |\psi_t\rangle\langle\psi_t|$ and $\hat{\Phi}(x) = \int d^3k \, |k|^{-1} \left(e^{2\pi ikx} a_k + e^{-2\pi ikx} a_k^*\right)$.

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Bogoliubov dynamics: $i\partial_t \chi_{\mathrm{B}}(t) = H^{\mathrm{B}}(t)\chi_{\mathrm{B}}(t)$ with initial datum $\chi_{\mathrm{B}}(0) = \chi$.

Here, $K(t,k,x) = \int d^3y\, q(t,x,y)\, |k|^{-1}\, e^{-2\pi iky}\psi_t(y)$ with $q(t) = 1 - |\psi_t\rangle\langle\psi_t|$ and $\hat{\Phi}(x) = \int d^3k\ |k|^{-1} \left(e^{2\pi ikx} a_k + e^{-2\pi ikx} a_k^*\right)$.

Norm approximation

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$$\Psi_N = W(\sqrt{N}\varphi) \sum_{k=0}^{N} \psi^{\otimes(N-k)} \otimes_s \chi^{(k)} \in \mathcal{H}^{(N)}$$

and $\Psi_{N,t} = e^{-iH_N^{\mathrm{F}} t} \Psi_N$. Then, there exists $C > 0$ such that

$$\left\| \Psi_{N,t} - \Psi_{N,t}^{\mathrm{B}} \right\|_{\mathcal{H}^{(N)}} \leq N^{-1/8} C e^{C|t|^9},$$

where $\Psi_{N,t}^{\mathrm{B}} = W(\sqrt{N}\varphi_t) \sum_{k=0}^{N} \psi_t^{\otimes(N-k)} \otimes_s \chi_{\mathrm{B}}^{(k)}(t)$.

Remarks

- Several norm approximations exist for systems with two-body interaction potentials:
  [Lewin, Nam, Schlein '15], [Boccato, Cenatiempo, Schlein '17], [Nam, Napiórkowski '17], [Brennecke, Nam, Napiórkowski, Schlein '19], [Nam, Salzmann '20] . . .
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Main difficulty: singular interaction

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Main difficulty: singular interaction

- energy estimates in the spirit of [LNS '15][2],
- truncation of the Bogoliubov dynamics[3],
- commutator method of [Lieb, Yamazaki '58],
- "refined version" of [Frank, Schlein '14].

[2] See also [Benedikter, de Oliviera, Schlein '15], [BCS '17], [NN'17], [Brennecke, Schlein '19] [BNNS '19], [NS '20].

[3] Similarly used in [Rodnianski, Schlein '09], [BNSS '19], [NN'17], [NS '20].

Sketch of the proof

Define $\mathcal{N} = \mathcal{N}_a + \mathcal{N}_b$ and $\chi(t) \in \mathcal{F} \otimes \mathcal{F}$ by $\chi^{(k)}(t) = \chi^{(k)}_{\leq N}(t)$ if $k \in \{1, 2, \ldots, N\}$ and zero else.

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Step 1:

$$\begin{aligned} a[\Psi_{N,t}, \psi_t, \varphi_t] &= N^{-1} \big\langle W^*(\sqrt{N}\varphi_t)\Psi_{N,t}, \mathcal{N}_a \, W^*(\sqrt{N}\varphi_t)\Psi_{N,t} \big\rangle \\ &\quad + \operatorname{Tr}_{L^2(\mathbb{R}^3)} \Big| \sqrt{1-\Delta}(1 - |\psi_t\rangle\langle\psi_t|)\gamma^{(1,0)}_{\Psi_{N,t}}(1 - |\psi_t\rangle\langle\psi_t|)\sqrt{1-\Delta} \Big| \\ &= N^{-1} \big\langle \chi(t), (\mathcal{N} + d\Gamma_b(-\Delta))\chi(t) \big\rangle. \end{aligned}$$

By

$$\begin{aligned} &\big\langle \chi(t), (\mathcal{N} + d\Gamma_b(-\Delta) + 1)\chi(t) \big\rangle \\ &\quad \leq C e^{C \int_0^t ds \, (\|\psi_s\|^2_{H^3} + \|\varphi_s\|^2_{L^2_2})} \big\langle \chi, (\mathcal{N} + d\Gamma_b(-\Delta) + 1)\chi \big\rangle. \end{aligned}$$

one obtains the result for the reduced densities.

Sketch of the proof

Step 2: Similarly, one can show

$$\begin{aligned}&\big\langle \chi_{\mathrm{B}}(t), (\mathcal{N} + d\Gamma_b(-\Delta) + 1)\chi_{\mathrm{B}}(t)\big\rangle \\ &\quad \leq C e^{C\int_0^t ds\,(\|\psi_s\|_{H^3}^2 + \|\varphi_s\|_{L_2^2}^2)} \big\langle \chi, (\mathcal{N} + d\Gamma_b(-\Delta) + 1)\chi\big\rangle\end{aligned}$$

but the growth of higher moments of the number operator is difficult to control. Note that

$$\frac{d}{dt}\|\chi(t) - \chi_{\mathrm{B}}(t)\|^2 = 2\,\mathrm{Im}\big\langle \chi(t), \big(H(t) - H^{\mathrm{B}}(t)\big)\,\chi_{\mathrm{B}}(t)\big\rangle$$

with

$$\begin{aligned}H(t) - H^{\mathrm{B}}(t) &= N^{-1/2}\int d^3x\, b_x^*\big(q(t)\hat{\Phi}q(t) - \big\langle \psi_t, \hat{\Phi}\psi_t\big\rangle\big)\, b_x \\ &\quad + \int d^3x \int d^3k\, K(t,k,x)\,(a_k^* + a_{-k})\, b_x^*\big(\big[1 - N^{-1}\mathcal{N}_b\big]_+^{1/2} - 1\big) + \mathrm{h.c.}\end{aligned}$$

Sketch of the proof

Step 3: We define

$$i\partial_t \chi_{\mathrm{B,M}}(t) = \mathbb{1}_{\mathcal{N}\leq M} H^{\mathrm{B}}(t) \mathbb{1}_{\mathcal{N}\leq M} \chi_{\mathrm{B,M}}(t) \quad \text{with} \quad \chi_{\mathrm{B,M}}(0) = \mathbb{1}_{\mathcal{N}\leq M}\chi.$$

For $\Lambda \geq 1$

$$\left\| (\mathcal{N}+1)^{\frac{k}{2}} \chi_{\mathrm{B,M}}(t) \right\|^2 \leq e^{C|t|} \left[\left\| (\mathcal{N}+1)^{\frac{k}{2}} \chi_{\mathrm{B,M}}(0) \right\|^2 \right.$$
$$\left. + \int_0^t ds \left(\Lambda \left\| (\mathcal{N}+1)^{\frac{k-1}{2}} \chi_{\mathrm{B,M}}(s) \right\|^2 + \Lambda^{-1} M^{k-1} \left\| (\mathcal{N} + d\Gamma_b(-\Delta) + 1)^{\frac{1}{2}} \chi_{\mathrm{B,M}}(t) \right\|^2 \right) \right]$$

holds. This implies

$$\sup \left\{ M^{-1/4} \left\| (\mathcal{N}+1)\chi_{\mathrm{B,M}}(t) \right\|, M^{-5/8} \left\| (\mathcal{N}+1)^{3/2} \chi_{\mathrm{B,M}}(t) \right\| \right\} \leq C e^{C|t|^9}.$$

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holds. This follows because of Gronwall's lemma and

$$\frac{d}{dt} \left\| (\mathcal{N}+1)^{\frac{k}{2}} \chi_{\mathrm{B,M}}(t) \right\|^2$$
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Note: 1) $\left((\mathcal{N}+3)^k - (\mathcal{N}+1)^k \right) \sim (\mathcal{N}+3)^{k-1}$.

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Note: 1) $(\mathcal{N}+3)^k - (\mathcal{N}+1)^k) \sim (\mathcal{N}+3)^{k-1}$.

2) $G_x(k) = |k|^{-1} e^{-2\pi i k\cdot x}$ and $(1+|k|^2) G_x(k) = G_x(k) + (2\pi)^{-1} [k \cdot i\nabla, G_x(k)]$.

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$$-2\mathrm{Im}\left\langle \chi_{\mathrm{B,M}}(t), ((\mathcal{N}+3)^k - (\mathcal{N}+1)^k) \int dx \int_{|k|\geq\Lambda} dk\, \overline{\psi_t(x)}\, \overline{G_x(k)} a_k b_x \chi_{\mathrm{B,M}}(t)\right\rangle$$

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holds. This follows because of Gronwall's lemma and

$$\frac{d}{dt} \left\| (\mathcal{N}+1)^{\frac{k}{2}} \chi_{\mathrm{B,M}}(t) \right\|^2$$

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$$\left\| \int_{|k|\leq\Lambda} dk\, G_x(k) a_k^* \psi_t(x) \chi \right\|^2 = \left\| \int_{|k|\leq\Lambda} dk\, \overline{G_x(k)} a_k \psi_t(x) \chi \right\|^2 + 4\pi\Lambda \left|\psi_t(x)\right|^2 \left\|\chi\right\|^2$$

$$\leq \left\| (1-\Delta_x)^{1/2} \psi_t(x) \mathcal{N}_a \chi \right\|^2 + 4\pi\Lambda \left|\psi_t(x)\right|^2 \left\|\chi\right\|^2.$$

Sketch of the proof

Step 4:

$$\left\|\Psi_{N,t}-\Psi^{\mathrm{B}}_{N,t}\right\|_{\mathcal{H}^{(N)}}=\left\|\left(\chi^{(k)}_{\leq N}(t)-\chi^{(k)}_{\mathrm{B}}(t)\right)_{k=0}^{N}\right\|_{\left(\bigoplus_{k=0}^{N}L^2_{\perp\psi_t}(\mathbb{R}^3)^{\otimes_s k}\right)\otimes\mathcal{F}}$$

$$\leq\|\chi(t)-\chi_{\mathrm{B}}(t)\|$$

$$\leq\|\chi(t)-\chi_{\mathrm{B,M}}(t)\|+\underbrace{\|\chi_{\mathrm{B,M}}(t)-\chi_{\mathrm{B}}(t)\|}_{\leq Ce^{C|t|^9}M^{-3/8}}.$$

Here,

$$\frac{d}{dt}\|\chi(t)-\chi_{\mathrm{B,M}}(t)\|^2=2\mathrm{Im}\left\langle\chi(t),\left(H(t)-\mathbb{1}_{\mathcal{N}\leq M}H^{\mathrm{B}}(t)\right)\mathbb{1}_{\mathcal{N}\leq M}\chi_{\mathrm{B,M}}(t)\right\rangle$$

with

$$H(t)-\mathbb{1}_{\mathcal{N}\leq M}H^{\mathrm{B}}(t)=\mathbb{1}_{\mathcal{N}>M}\left(\int d^3x\,b_x^*h_{\varphi_t}b_x+\mathcal{N}_a\right)$$

$$+\mathbb{1}_{\mathcal{N}>M}\int d^3x\int d^3k\,K(t,k,x)\,(a_k^*+a_{-k})\,b_x^*\left[1-N^{-1}\mathcal{N}_b\right]_+^{1/2}+\text{h.c.}$$

$$+\mathbb{1}_{\mathcal{N}\leq M}\int d^3x\int d^3k\,K(t,k,x)\,(a_k^*+a_{-k})\,b_x^*\left(\left[1-N^{-1}\mathcal{N}_b\right]_+^{1/2}-1\right)+\text{h.c.}$$

$$+N^{-1/2}\int d^3x\,b_x^*\left(q(t)\hat{\Phi}q(t)-\langle\psi_t,\hat{\Phi}\psi_t\rangle\right)b_x.$$

Strong coupling limit

$$i\partial_t \Psi_t = H^{\mathrm{F}}\,\Psi_t,$$

$$H^{\mathrm{F}} = -\Delta + \sqrt{\alpha}\int d^3k\,\left(G_x(k)a_k^* + \overline{G_x(k)}a_k\right) + \int d^3k\; a_k^* a_k,$$

$$[a_k, a_l^*] = \delta(k-l) \quad \text{and} \quad [a_k, a_l] = [a_k^*, a_l^*] = 0.$$

Note: $N = 1$ and $\mathcal{H} = L^2(\mathbb{R}^3) \otimes \left[\oplus_{n\geq 0} L^2(\mathbb{R}^3)^{\otimes_s^n}\right]$.

Strong coupling limit

$$\alpha^2 i\partial_t \Psi_t = H^{\mathrm{F}}\, \Psi_t,$$

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$$\begin{cases} i\partial_t \psi_t(x) & = \left[-\Delta + \Phi_{\varphi_t}(x)\right] \psi_t(x) \\ i\partial_t \alpha^2 \varphi_t(k) & = \varphi_t(k) + |k|^{-1} \int d^3x\, e^{-2\pi i k\cdot x}\, |\psi_t(x)|^2 \end{cases}. \qquad \text{(LP-2)}$$

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Let $\varphi_0 \in L^2(\mathbb{R}^3)$ such that $e(\varphi_0) = \inf_\psi \langle \psi, (-\Delta + \Phi_{\varphi_0})\, \psi \rangle < 0$, ψ_{φ_0} be the g.s. of $-\Delta + \Phi_{\varphi_0}$ and (ψ_t, φ_t) be the solution of (LP-2) with i.d. $(\psi_{\varphi_0}, \varphi_0)$.

[L., Mitrouskas, Rademacher, Schlein, Seiringer '21]: Let $\langle \Upsilon, \mathcal{N}_a^5 \Upsilon \rangle_{\mathcal{F}} \leq c\alpha^{-10}$. Then, there exist $C, T > 0$ such that for all $|t| \leq T\alpha^2$

$$\left\| e^{-iH^{\mathrm{F}} t} \left(\psi_{\varphi_0} \otimes W(\alpha^2 \varphi_0) \Upsilon \right) - e^{-i \int_0^t du\, \omega(u)} \psi_t \otimes W(\alpha^2 \varphi_t) \Upsilon_t \right\|^2 \leq C\alpha^{-1}.$$

Note that $\Upsilon \in \mathcal{F}$ satisfies $i\partial_t \Upsilon_t = (\mathcal{N}_a - \mathcal{A}_t)\, \Upsilon_t$ with $\Upsilon_0 = \Upsilon$. Here, $\mathcal{A}_t = \langle \psi_{\varphi_t}, \phi(G_\cdot)\, R_{\varphi_t} \phi(G_\cdot)\, \psi_{\varphi_t} \rangle_{L^2(\mathbb{R}^3)}$ with $q_{\varphi_t} = 1 - |\psi_{\varphi_t}\rangle\langle\psi_{\varphi_t}|$ and $R_t = q_{\varphi_t} \left(h_{\varphi_t} - e(\varphi_t) \right) q_{\varphi_t}$.

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Remarks:

- The L.P.-equations without quantum fluctuations approximate well the time evolution of one-particle reduced density matrices.

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Remarks:

- The L.P.-equations without quantum fluctuations approximate well the time evolution of one-particle reduced density matrices.
- previous results: [Frank, Schlein '14], [Frank, Gang '15], [Griesemer '17], [L., Rademacher, Schlein, Seiringer '19], [Mitrouskas '20] and [Frank, Gang'19] (adiabatic theorem in 1d).

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- We have considered norm approximations for the Fröhlich dynamics in the mean-field and strong coupling regime.

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Summary

- We have considered norm approximations for the Fröhlich dynamics in the mean-field and strong coupling regime.
- In the mean-field regime quantum fluctuations among and between the particles and the phonons must be taken into account whereas it is sufficient to consider only correlations between the phonons in the strong coupling limit.

Thank you for your attention!

Related references

Many-body mean-field limits:

[Falconi '12], [Ammari, Falconi '17], [L., Pickl '18], [L., Pickl '20], [L., Petrat '19], [L., Mitrouskas, Seiringer '21], [Falconi, L., Mitrouskas, Petrat '21], [L' 22].

Strong coupling limit:

[Frank, Schlein '14], [Frank, Gang '17], [Griesemer '17], [L., Rademacher, Schlein, Seiringer '19], [Mitrouskas '21], [L., Mitrouskas, Rademacher, Schlein, Seiringer '21], [Feliciangeli, Rademacher, Seiringer '21].

Partially classical limit:

[Ginibre, Nironi, Velo '06], [Correggi, Falconi '18], [Correggi, Falconi, Olivieri '19], [Carlone, Correggi, Falconi, Olivieri '20], [Correggi, Falconi, Olivieri '21].

Classical limit:

[Knowles '09], [Ammari, Falconi, Hiroshima '22].

Approximation of the quantum field by a two-particle potential:

[Davies '79], [Hiroshima '98], [Teufel '02].

Moreover, note that there are recent results about the ground state energy and effective mass of the polaron.