Correlation energy of mean-field Fermi gases

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Summary

- Introduction: many-body Fermi systems, mean-field regime.
- Effective theories for mean-field Fermi gases: Hartree-Fock theory.
- Main result: correlation energy for mean-field Fermi gases.
- Ideas of the proof: Fock space representation, Bogoliubov transformations, collective bosonization.
- Norm approximation for many-body quantum dynamics (if time permits)
- Conclusions.

Introduction

Many-body Hamiltonian

- We consider a system of $N \gg 1$ fermions, confined in $\Lambda = \mathbb{T}^3$.
  State of the system: $\psi_N \in L^2_{\mathrm{a}}(\mathbb{T}^{3N})$.

- Many-body Hamiltonian:

$$H_N = \sum_{j=1}^{N} -\Delta_j + \lambda \sum_{i<j}^{N} V(x_i - x_j) \qquad \text{on } L^2(\mathbb{T}^{3N}).$$

  Mean field regime. The potential is N-indep., hence every particle interacts with a macroscopic fraction of the others. One expects:

$$\Big\langle \psi_N, \lambda \sum_{i<j}^{N} V(x_i - x_j)\psi_N \Big\rangle \sim \lambda N^2 \ .$$

- We shall choose $\lambda \equiv \lambda(N)$ so that kinetic and potential energy are of the same order in N. How large is the kinetic energy?

The free Fermi gas

- For $p \in \mathbb{Z}^3$, let $f_p(x)$ be the eigenstates of $-\Delta$ on $L^2(\mathbb{T}^3)$ (plane waves):

$$f_p(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{ip\cdot x} , \qquad -\Delta f_p = |p|^2 f_p \; .$$

  The functions $(f_p)_{p\in\mathbb{Z}^3}$ provide an orthonormal basis for $L^2(\mathbb{T}^3)$.

- An ONB for $L^2_{\mathrm{a}}(\mathbb{T}^{3N})$ is provided by the Slater determinants, constructed using the plane waves:

$$\psi_N = f_{p_1} \wedge \ldots \wedge f_{p_N} \; .$$

  Notice that $\psi_N \equiv 0$, if $p_i = p_j$ for $i \neq j$ (Pauli principle).

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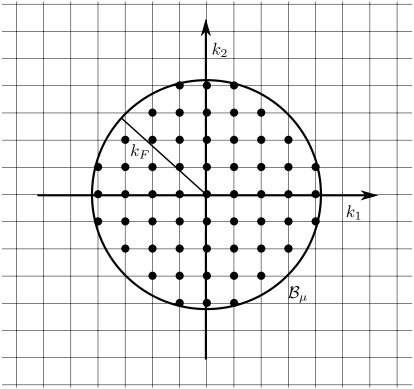
- The kinetic energy of a Slater determinant is:

$$\Big\langle \psi_N, \sum_{i=1}^{N} -\Delta_i \psi_N \Big\rangle = \sum_{p\in\{p_1,\ldots,p_N\}} |p|^2 .$$

Fermi ball

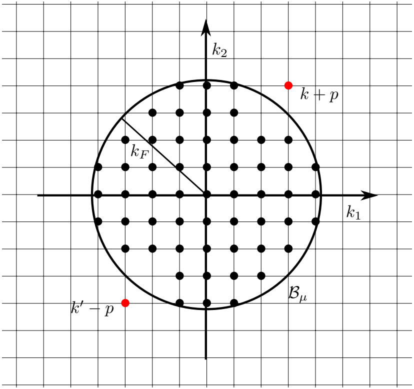
- The ground state vector is obtained filling a ball $\mathcal{B}_\mu$ in $\mathbb{Z}^3$:



- Black dots correspond to occupied momentum states in $f_{p_1} \wedge \ldots \wedge f_{p_N}$. From now on, $N \equiv N(k_F)$ so that Fermi ball is completely filled.

Excited states

- Excitations: holes in the Fermi sea.



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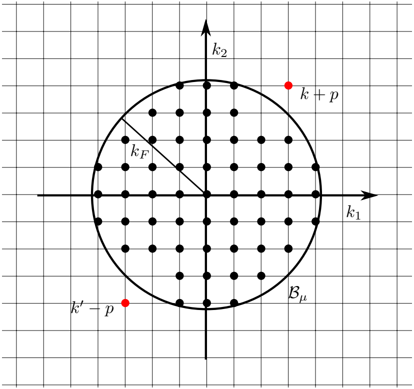

Rmk: if $V(x_i - x_j) \neq 0$ product states are no longer eigenstates of H_N.
Correlations among the N particles destroy the product structure.

Mean-field Hamiltonian

- Ground state energy of the noninteracting system, for $N \to \infty$:

$$\sum_{p:|p|\leq cN^{\frac{1}{3}}} |p|^2 \simeq N^{\frac{5}{3}} \int_{|p|\leq c} dp\, |p|^2 = O(N^{\frac{5}{3}})$$

More generally, Lieb-Thirring kinetic energy inequality:

$$\Big\langle \psi_N, \sum_{i=1}^{N} -\Delta_i \psi_N \Big\rangle \geq C_{\rm LT} \int dx\, \rho_{\psi_N}^{\frac{5}{3}}(x)\;, \qquad \forall \psi_N \in L^2_{\rm a}(\mathbb{R}^{3N})$$

with $\rho_{\psi_N}(x) = N \int dx_2 \ldots dx_N\, |\psi_N(x, x_2, \ldots, x_N)|^2$.

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- Mean-field Hamiltonian:

$$H_N = \sum_{j=1}^{N} -\varepsilon^2 \Delta_j + \frac{1}{N} \sum_{i<j}^{N} V(x_i - x_j) \qquad \text{with } \varepsilon = N^{-1/3}\,.$$

We shall be interested in the ground state energy of the system:

$$E_N := \inf_{\psi_N \in L^2_{\mathrm{a}}(\mathbb{T}^{3N}):\, \|\psi_N\|=1} \langle \psi_N, H_N \psi_N \rangle$$

Hartree-Fock theory and the correlation energy

Hartree-Fock theory

- In the mean-field regime, one expects every particle to be subject to an "average field", generated by all the others:

$$\langle\psi_N, \sum_{i<j}^N V(x_i - x_j)\psi_N\rangle \simeq \frac{1}{2}\langle\psi_N, \sum_{i=1}^N (V * \rho_{\psi_N})(x_i)\psi_N\rangle$$

  Reasonable approximation, provided the correlations among the particles are not too strong (law of large numbers).

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- Slater determinants are the most uncorrelated fermionic states: the only correlation among the particles is the one due to antisymmetry:

$$\psi_N = f_1 \wedge \cdots \wedge f_N \ , \qquad \langle f_i, f_j\rangle = \delta_{ij} \ .$$

- Hartree-Fock approximation. Minimize H_N over uncorrelated states. The Hartree-Fock ground state energy is defined as:

$$E_N^{\mathrm{HF}} := \inf_{\psi_N \text{ Slater}} \langle\psi_N, H_N\psi_N\rangle$$

Hartree-Fock energy functional

- Given a fermionic state ψ_N, the reduced k-particle density matrix is:

$$\gamma_N^{(k)} := \binom{N}{k} \operatorname{tr}_{k+1 \to N} |\psi_N\rangle\langle\psi_N| \,.$$

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- For a Slater determinant, $f_1 \wedge \ldots \wedge f_N$, f_i orthonormal, the reduced 1PDM takes a particularly simple form:

$$\gamma_N^{(1)} = \sum_{i=1}^{N} |f_i\rangle\langle f_i| =: \omega_N \,.$$

Notice that $\omega_N = \omega_N^2 = \omega_N^*$, $\operatorname{tr}\omega_N = N$.

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$$\gamma_N^{(k)}(x_1, \ldots, x_k; y_1, \ldots, y_k) = \frac{1}{k!} \sum_{\pi} \sigma_\pi \prod_{j=1}^{k} \omega(x_j; y_{\pi(j)}) \, .$$

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- Energy of a Slater determinant: Hartree-Fock energy functional,

$$\mathcal{E}_N^{\mathrm{HF}}(\omega_N) = -\operatorname{tr}\varepsilon^2\Delta\omega_N + \frac{1}{2N}\int dxdy\, V(x-y)[\rho_\omega(x)\rho_\omega(y) - |\omega_N(x;y)|^2]$$

with $\rho_\omega(x) = \omega_N(x;x)$.

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$$E_N \geq E_N^{\mathrm{HF}} - CN^{\frac{1}{3}-\delta} \qquad \text{for some } \delta > 0$$

The result allows to resolve the full Hartree-Fock energy, since the size of the exchange term, the smallest term in the functional, is $O(N^{\frac{1}{3}})$.

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- Related (and earlier) results: comparison with Thomas-Fermi theory. In the TF approximation, the energy of a fermionic state is:

$$\mathcal{E}_N^{\mathrm{TF}}(\rho) = \frac{3}{5}(6\pi^2)^{\frac{2}{3}}\varepsilon^2 \int dx\, \rho(x)^{\frac{5}{3}} + \frac{1}{2N}\int V(x-y)\rho(x)\rho(y)$$

  It arises as a semiclassical approximation of quantum mechanics. [Lieb-Simon '77; ... ; Fournais-Lewin-Solovej '15.]

  The TF ground state energy is the leading order of the HF energy.

The correlation energy

- Main limitation: HF and TF theories completely neglect the effect of correlations (the true ground state is not a Slater determinant!).
- In our confined setting the HF ground state is the free Fermi gas:

$$E_N^{\mathrm{HF}} = \mathcal{E}_N^{\mathrm{HF}}(\omega_N)\,, \qquad \omega_N = \sum_{p \in \mathcal{B}_\mu} |f_p\rangle\langle f_p|\,.$$

  Reason: spectral gap of the Laplacian. In the infinite volume limit, translation invariance might be broken. [Gontier-Hainzl-Lewin '18]

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- We define the correlation energy as:

$$E_N^{\mathrm{C}} := E_N - E_N^{\mathrm{HF}} \;.$$

  Attempts for the computation of E_N^{C} have been made since the early days of condensed matter physics: Wigner '34, Heisenberg '47, Macke '50, Bohm-Pines '53, Sawada '57, Gell-Mann and Brueckner '57...

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- Bohm-Pines: Random-phase approximation. Infinite resummation of a suitable class of Feynman graphs.

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Theorem (BNPSS CMP '20, Inventiones '21; BPSS arXiv 2021)

Let $\hat{V}(k) \geq 0$, $|k|\hat{V}(k) \in \ell^1(\mathbb{Z}^3)$. *Then, as* $N \to \infty$*:*

$$E_N^C = \varepsilon\kappa \sum_{k\in\mathbb{Z}^3} |k| \left(\frac{1}{\pi} \int_0^\infty \log\left[1 + 2\pi\kappa\hat{V}(k)\left(1 - \lambda \arctan\left(\lambda^{-1}\right)\right)\right] d\lambda - \frac{\pi}{2}\kappa\hat{V}(k) \right) + o(\varepsilon)$$

where $\kappa = (3/4\pi)^{\frac{1}{3}}$, $\varepsilon = N^{-\frac{1}{3}}$.

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where $\kappa = (3/4\pi)^{\frac{1}{3}}$, $\varepsilon = N^{-\frac{1}{3}}$.

Remarks.

- The result agrees with the random-phase approximation of Bohm-Pines.
- Previous work, at second order in V: Hainzl, P., Rexze '19.
  Correlation energy at all orders, for small V: BNPSS '20-'21
- Result based on collective bosonization, for low energy excitation.
  Related approach: Christiansen, Hainzl, Nam '21.
- As an upper bound, the theorem extends to $|k|\hat{V}(k)^2 \in \ell^1(\mathbb{Z}^3)$.

Sketch of the proof

Fock space formulation

- Starting point: formulation of the problem in the fermionic Fock space,

$$\mathcal{F} = \bigoplus_{n\in\mathbb{N}} L^2_{\mathrm{a}}(\mathbb{T}^{3N}) = \mathbb{C} \oplus L^2(\mathbb{T}) \oplus L^2_{\mathrm{a}}(\mathbb{T}^2) \oplus \dots .$$

Vacuum vector: $\Omega = (1, 0, 0, \dots, 0, \dots)$.

- Useful to introduce fermionic creation and annihilation operators:

$$a_k : \mathcal{F}^{(n)} \to \mathcal{F}^{(n-1)} , \qquad a^*_k : \mathcal{F}^{(n)} \to \mathcal{F}^{(n+1)}$$

Explicitly, for any $\psi \in \mathcal{F}$, and for any $k \in \mathbb{Z}$:

$$\begin{aligned}(a_k\psi)^{(n)}(x_1, \dots, x_n) &= \sqrt{n+1} \int_{\mathbb{T}} dx\, f_k(x)\, \psi^{(n+1)}(x, x_1, \dots, x_n) \\ (a^*_k\psi)^{(n)}(x_1, \dots, x_n) &= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (-1)^j f_k(x_j) \psi^{(n-1)}(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)\end{aligned}$$

$(a_k\Omega = 0)$. Canonical anticommutation relations:

$$\{a_k, a_{k'}\} = \{a^*_k, a^*_{k'}\} = 0 , \qquad \{a^*_k, a_{k'}\} = \delta_{k,k'} .$$

Fock space Hamiltonian

- Creation/annihilation operators can be used to lift operators from $L^2_{\mathrm{a}}(\mathbb{T}^{3n})$ to the Fock space. Simplest example: number operator,

$$\mathcal{N} = \bigoplus_{n \in \mathbb{N}} n \mathbf{1}_{L^2_{\mathrm{a}}(\mathbb{T}^{3n})} = \sum_{k \in \mathbb{Z}^3} a^*_k a_k \ .$$

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- Fock space Hamiltonian: $\mathcal{H}_N = \bigoplus_n H_N^{(n)}$, with $H_N^{(n)}$ on $L^2_{\mathrm{a}}(\mathbb{T}^{3n})$. Equivalently,

$$\mathcal{H}_N = \sum_k \varepsilon^2 |k|^2 a_k^* a_k + \frac{1}{2N} \sum_{p,k,k'} \hat{V}(p) a^*_{k+p} a^*_{k'-p} a_{k'} a_k$$

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- The ground state energy E_N of $H_N \equiv H_N^{(N)}$ on $L^2_{\rm a}(\mathbb{T}^{3N})$ is:

$$E_N = \inf_{\psi\in\mathcal{F}^{(N)}} \frac{\langle \psi, \mathcal{H}_N \psi\rangle_{\mathcal{F}}}{\langle \psi,\psi\rangle_{\mathcal{F}}}\;,$$

with $\langle \psi,\varphi\rangle_{\mathcal{F}} = \sum_n \langle \psi^{(n)}, \varphi^{(n)}\rangle_{L^2(\mathbb{T}^{3n})}$.

Bogoliubov transformations

- It turns out that $\mathcal{F}$ can be built acting repeatedly with the $\{a_k^*\}$ operators on the vacuum vector $\Omega = (1, 0, \ldots, 0, \ldots)$:

$$(a_{k_1}^* \cdots a_{k_n}^* \Omega)^{(n)}(x_1, \ldots, x_n) = \frac{1}{\sqrt{n!}} \Big(\det f_{k_i}(x_j) \Big)_{1 \le i,j \le n}$$

and Slater dets form a basis of $L^2_{\mathrm{a}}(\mathbb{T}^{3n})$. Each a_k^* adds a "quantum" of kinetic energy $\varepsilon^2 |k|^2$ on the state (the energy of Ω is zero).

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- Bogoliubov transformation: convenient rotation of the vacuum vector. There exists a unitary operator $R : \mathcal{F} \to \mathcal{F}$, such that:

$$R^* a_k^* R = \begin{cases} a_k^* & k \notin \mathcal{B}_\mu \quad \text{(creates a particle)} \\ a_k & k \in \mathcal{B}_\mu \quad \text{(creates a hole)} \end{cases} , \qquad R\Omega = \bigwedge_{k \in \mathcal{B}_\mu} f_k \,.$$

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- Advantage: the new vacuum $R\Omega$ is the Hartree-Fock ground state:

$$\langle R\Omega, \mathcal{H}_N R\Omega \rangle = E_N^{\mathrm{HF}} \,.$$

The correlation Hamiltonian

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  We have, highlighting only the most relevant terms:

$$R^*\mathcal{H}_N R \;=\; E_N^{\mathrm{HF}} + \mathbb{H}_0 + \mathbb{Q} + \mathbb{Q}^* + \mathbb{E}\,,$$

where: $\qquad (\mu = \varepsilon^2 k_F^2)$

$$\mathbb{H}_0 \;=\; \sum_k e(k) a_k^* a_k\,, \qquad e(k) = |\varepsilon^2 |k|^2 - \mu| \qquad \text{(relative kin. energy)}$$

$$\mathbb{Q} \;=\; \frac{1}{N} \sum_{\substack{k,k' \in \mathcal{B}_\mu \\ k+p, k'-p \notin \mathcal{B}_\mu}} \hat{V}(p) a_k a_{k+p} a_{k'} a_{k'-p} \qquad \text{(excitations around } \partial B_\mu\text{)}$$

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- $\mathcal{H}_N^{\mathrm{C}} := \mathbb{H}_0 + \mathbb{Q} + \mathbb{Q}^* + \mathbb{E}$ is the correlation Hamiltonian.
  The correlation energy can be rewritten as:
  $$E_N^{\mathrm{C}} = \inf_{\xi\in R^*\mathcal{F}^{(N)}} \langle \xi, \mathcal{H}_N^{\mathrm{C}} \xi\rangle\,.$$

Global pseudo-bosons

- The Bogoliubov-transformed interaction $\mathbb{Q}$ can be rewritten as:

$$\mathbb{Q} = \frac{1}{N} \sum_{p} \hat{V}(p) b_p b_{-p} \;, \qquad b_p = \sum_{\substack{k \in \mathcal{B}_\mu \\ k+p \notin \mathcal{B}_\mu}} a_{k+p} a_k \;.$$

The b_p operators implement particle-hole excitations.

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- The Bogoliubov-transformed interaction $\mathbb{Q}$ can be rewritten as:

$$\mathbb{Q} = \frac{1}{N} \sum_{p} \hat{V}(p) b_p b_{-p} \; , \qquad b_p = \sum_{\substack{k \in \mathcal{B}_\mu \\ k+p \notin \mathcal{B}_\mu}} a_{k+p} a_k \; .$$

The b_p operators implement particle-hole excitations. They behave as bosons, if evaluated on states with "few" particles:

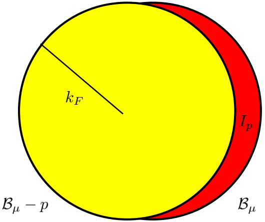
$$[b_p, b_q] = [b_p^*, b_q^*] = 0 \; , \qquad \langle \xi, [b_p, b_q^*] \xi \rangle \simeq \delta_{p,q} n_p^2 + \langle \xi, \mathcal{N} \xi \rangle$$

with (see figure):

(i) n_p^2 = number of lattice points in the red region I_p,

(ii) $k_F = O(N^{1/3}) \Rightarrow n_p^2 \sim |p| N^{2/3}$.

Approximate CCR if $\langle \xi, \mathcal{N} \xi \rangle \ll N^{2/3}$.



Bosonization of the kinetic energy

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$$\mathbb{H}_0 b_p^* \Omega = [\mathbb{H}_0, b_p^*]\Omega = \sum_{k \in I_p} \varepsilon^2 |p \cdot k| \, a_{k+p}^* a_k^* \Omega \, .$$

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- This is precisely what happens in the Luttinger model, where the dispersion relation is linear. There:

$$(p \cdot k) \to \pm |p| k_F \, , \qquad \pm\text{: chirality of the fermions.}$$

  Mattis-Lieb '65: exact solution of the Luttinger model in terms of the excitations of a noninteracting Bose gas. First instance of bosonization.

Local pseudo-bosons

- Idea: localize particle-hole excitations in patches on the Fermi surface, and there linearize. [Haldane-Luther 90s; Benfatto-Gallavotti 90s]

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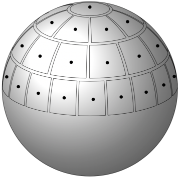

(i) $c_{p,\alpha} = (n_{p,\alpha})^{-1} \sum_{k\in I_p \cap P_\alpha} a_{k+p} a_k$

$\#(P_\alpha) = M \gg 1, \quad n_{p,\alpha}^2 \simeq |p\cdot\hat{\omega}_\alpha| \frac{N^{2/3}}{M}$

(ii) Approximate CCR:

$$[c_{p,\alpha}, c_{q,\beta}] = 0\ , \qquad \langle \xi, [c_{q,\beta}, c^*_{p,\alpha}]\xi\rangle \simeq \delta_{\alpha,\beta}\Big(\delta_{p,q} + \frac{M}{N^{2/3}}\langle\xi,\mathcal{N}\xi\rangle\Big)$$

Almost-bosonic relations if $\langle\xi,\mathcal{N}\xi\rangle \ll M^{-1}N^{2/3}$.

(iii) On states with "few bosons" $\xi = \prod_i c^*_{\alpha_i,p_i}\Omega, \quad \omega_\alpha = \text{center}(P_\alpha)$:

$$\mathbb{H}_0\xi \simeq \Big(\sum_{p,\alpha} \varepsilon|p\cdot\hat{\omega}_\alpha| c^*_{p,\alpha}c_{p,\alpha}\Big)\xi \equiv \mathbb{D}_\mathrm{B}\xi \qquad (\mathbb{D}_\mathrm{B}\ \text{bosonic kin. en.})$$

Bosonization of the correlation Hamiltonian

- On states with few bosonic excitations, $\mathcal{H}_N^{\mathrm{C}}$ acts similarly to:

$$
\begin{aligned}
\mathcal{H}_N^{\mathrm{B}} &= \sum_{p,\alpha} \varepsilon |p \cdot \hat{\omega}_\alpha| c^*_{p,\alpha} c_{p,\alpha} \\
&\quad + \frac{1}{2N} \sum_{p,\alpha,\beta} \hat{V}(p) [n_{\alpha,p} n_{\beta,p} c^*_{p,\alpha} c_{p,\beta} + n_{\alpha,p} n_{\beta,-p} c^*_{p,\alpha} c^*_{-p,\beta} + \text{h.c.}]
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- If the theory was truly bosonic, the ground state would have the form:

$$\xi = T\Omega\,, \qquad T = \exp\Big\{ \sum_{\alpha,\beta,p} K_{\alpha,\beta}(p) c_{p,\alpha} c_{-p,\beta} - \mathrm{h.c.} \Big\}$$

for a suitable Bogoliubov kernel K. We would have:

$$T^* \mathcal{H}_N^{\mathrm{B}} T = E_N^{\mathrm{RPA}} + \mathbb{H}^{\mathrm{exc}}$$

with E_N^{RPA} the expression in our theorem and $\mathbb{H}^{\mathrm{exc}} \geq 0$ (excitations).

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- Take $\xi = T\Omega$ as a fermionic trial state. We have $\langle \xi, \mathcal{N}^k \xi\rangle \leq C_k$. Approximate bosonization is justified, and we get $E_N^{\mathrm{C}} \leq E_N^{\mathrm{RPA}} + o(\varepsilon)$.

Lower bound: a priori estimates

- Let V continuous, $\hat{V}(k) \geq 0$. The following inequality holds:

$$\sum_{i<j}^{N} V(x_i - x_j) \geq \frac{N^2}{2}\hat{V}(0) - \frac{N}{2}V(0)\ .$$

The first term is equal to the direct energy in E_N^{HF}. The second term is a semiclassical approximation for the exchange energy:

$$-\frac{1}{2N}\int dxdy\, V(x-y)|\omega_N(x;y)|^2 = -\frac{V(0)}{2} + O(\varepsilon)\ .$$

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- Let $\psi \in \mathcal{F}^{(N)}$ such that $E(\psi) \leq E_N^{\mathrm{HF}}$.
  Let $\xi = R^*\psi$. Using that $\langle \psi, \mathcal{K}\psi\rangle = -\operatorname{tr}\varepsilon^2\Delta\omega_N + \langle \xi, \mathbb{H}_0\xi\rangle$, we easily get:

$$E_N^{\mathrm{HF}} \geq E_N^{\mathrm{HF}} + \langle \xi, \mathbb{H}_0\xi\rangle - C\varepsilon\;.$$

- A priori estimates: $\langle \xi, \mathbb{H}_0\xi\rangle \leq C\varepsilon$, which also implies $\langle \xi, \mathcal{N}\xi\rangle \leq CN^{1/3}$.

Non-bosonizable terms

- The a priori estimate allow to use approximate bosonization.
- However, many terms in the correlation Hamiltonian are not bosonizable! Example:

$$\mathcal{E}_1 = \frac{1}{N} \sum_p \hat{V}(p) D_p^* D_p \ , \qquad D_p = \sum_{k:\, k+p, k \notin \mathcal{B}_\mu} a_{k+p}^* a_k - \sum_{k:\, k-p, k \in \mathcal{B}_\mu} a_{k-p}^* a_k$$

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$$\langle \xi, \mathcal{E}_1 \xi\rangle \le \frac{C}{N}\langle \xi, \mathcal{N}^2_{\frac{1}{3}-\delta}\xi\rangle\,, \qquad \mathcal{N}_{\frac{1}{3}-\delta} := \sum_{\mathrm{dist}(k,\mathcal{B}_\mu)\ge N^{-\frac{1}{3}+\delta}} a_k^* a_k$$

Using that $\mathcal{N}_{\frac{1}{3}-\delta} \le CN^{\frac{2}{3}-\delta}\mathbb{H}_0$, we would then get $\langle \xi, \mathcal{E}_1 \xi\rangle = o(\varepsilon)$.

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- Gapless modes. Their number is much smaller than $N^{2/3}$, thanks to refined lattice counting arguments (which I will not discuss).

Bosonization of the kinetic energy

- In order to compare $\mathbb{H}_0$ with $\mathbb{D}_\mathrm{B}$, we write:

$$\langle \xi, \mathbb{H}_0 \xi \rangle = \langle \xi, \mathbb{D}_\mathrm{B} \xi \rangle + \langle \xi, (\mathbb{H}_0 - \mathbb{D}_\mathrm{B}) \xi \rangle$$

The first term contributes to the correlation energy. The second term almost-commutes with the $c_{p,\alpha}$ operators. Hence:

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- Lower bound. Recall that, by approximate bosonization:

$$\langle \xi, (\mathbb{D}_{\mathrm{B}} + \mathbb{Q}) \xi \rangle \simeq E_N^{\mathrm{RPA}} + \langle T^* \xi, \mathbb{H}^{\mathrm{exc}} T^* \xi \rangle$$

The excitation Hamiltonian $\mathbb{H}^{\mathrm{exc}}$ is positive. Up to a new Bogoliubov transformation Z, it can be used to control $-\mathbb{D}_{\mathrm{B}}$:

$$Z^* \mathbb{H}^{\mathrm{exc}} Z \geq \mathbb{D}_{\mathrm{B}} \qquad \text{(exact if the operators were truly bosonic.)}$$

Norm approximation of quantum dynamics

Excited states?

- The previous analysis gives a rather precise understanding of the low-energy states of our mean-field problem.
- Natural guess: describe excited states in terms of

$$\Psi(\varphi_1; \ldots ; \varphi_m) := RTc^*(\varphi_1) \cdots c^*(\varphi_m)\Omega\,, \quad \text{with } \varphi_i \equiv \varphi_i(p, \alpha)$$

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- More precisely, one would like to argue that, for long times:

$$e^{-i\mathcal{H}_N t/\varepsilon}\Psi(\varphi_1; \ldots ; \varphi_m) \simeq e^{-i(t/\varepsilon)(E_N^{\mathrm{HF}}+E_N^{\mathrm{RPA}})}\Psi(\varphi_{1,t}; \ldots ; \varphi_{m,t})$$

  where $\varphi_{i,t} = e^{-iH_{\mathrm{B}}^{\mathrm{exc}} t/\varepsilon}\varphi_i$.

Norm approximation for quantum dynamics

Theorem (BNPSS, Annales Henri Poincaré 2021)

Let $\hat{V}(k)$ be compactly supported and nonnegative. Then, for any $m \in \mathbb{N}$ and for any $t \in \mathbb{R}$:

$$\left\| e^{-i\mathcal{H}_N t/\varepsilon}\Psi(\varphi_1;\ldots;\varphi_m) - e^{-i(t/\varepsilon)(E_N^{HF}+E_N^{RPA})}\Psi(\varphi_{1,t};\ldots;\varphi_{m,t}) \right\| \leq C_m \varepsilon^{\frac{1}{15}} |t| \,.$$

Remarks.

- The macroscopic time scale is $t = O(1)$. In fact, in our semiclassical scaling, the typical velocity of the particles is $O(1)$.
- The vector $\Psi(\cdot)$ is an N-particle state, and convergence holds in the $L^2(\mathbb{T}^{3N})$-norm.
- First result about norm-approximation of many-body quantum dynamics in terms of an effective dynamics. Previous convergence results, at the level of density matrices:

  Benedikter, P., Schlein '14 ; + Jaksic, Saffirio '16; Saffirio *et al.* '21.

Conclusions

- We rigorously computed the leading order in N of the correlation energy for fermionic systems in the mean-field regime, nonperturbatively.
- Proof based on rigorous bosonization. It allows to justify the Random Phase Approximation of Bohm-Pines, for the ground state energy.
- The method can be used to prove a norm approximation for the many-body evolution of a class of states, in terms of a simpler effective dynamics for the excitations around the Fermi surface.
- Similar ideas (patch-free) work in the completely different setting of dilute Fermi gases. They allow to understand the ground state energy as the energy of a quasi-free Bose gas [Falconi, Giacomelli, Hainzl, P. '21.]

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- Extension to Coulomb interactions?
- High density/thermodynamic limit?
- Superconducting instability?