Non-local Luttinger model out of equilibrium: Exact results and emergence of generalized hydrodynamics

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Outline

- Closed 1+1D systems out of equilibrium
- Non-local Luttinger model
- Exact analytical transport results
- Generalized hydrodynamics
- Emergence of hydrodynamics

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Partitioning protocol

Example: Harmonic chain

E.g.: Lebowitz, Spohn, CMP (1977)

$$H = \sum_j \left[\frac{p_j^2}{2m} + V(q_{j+1} - q_{j+1}) \right] \qquad V(q) \propto q^2$$

Fig.: Brun, Hartle, PRD (1999)

Example: XY spin chain

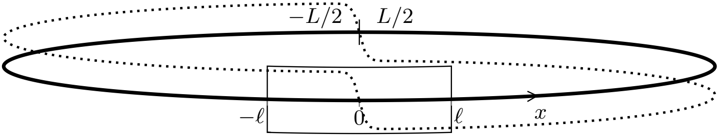
E.g.: Aschbacher, Pillet, JSP (2003)

$$H = -J \sum_j \left[(1+\gamma) S_j^x S_{j+1}^x + (1-\gamma) S_j^y S_{j+1}^y \right] \qquad |\gamma| \leq 1$$

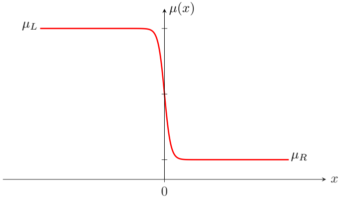
Initial state: Temperatures β_L^{-1} for $j \leq M-1$ and β_R^{-1} for $j \geq N+1$

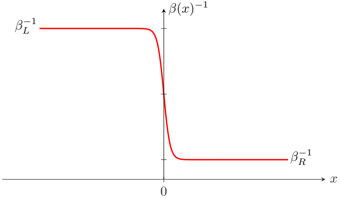
Smooth profiles

Continuum model on the circle



Initial state $\frac{\mathrm{e}^{-G}}{\mathrm{Tr}[\mathrm{e}^{-G}]}$ with $G = \int_{-L/2}^{L/2} \mathrm{d}x\, \beta(x)\big[\mathcal{E}(x) - \mu(x)\rho(x)\big]$

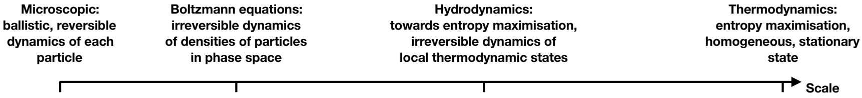




Consider $\frac{\mathrm{Tr}[\mathrm{e}^{-G}\mathcal{O}(x,t)]}{\mathrm{Tr}[\mathrm{e}^{-G}]}$ for local observables $\mathcal{O}(x,t) = \mathrm{e}^{\mathrm{i}Ht}\mathcal{O}(x)\mathrm{e}^{-\mathrm{i}Ht}$

Hydrodynamic description

Reduction of degrees of freedom:



Doyon, SciPost Phys. Lect. Notes (2020)

Generalized hydrodynamics (GHD)

Castro-Alvaredo, Doyon, Yoshimura, PRX (2016)
Bertini, Collura, De Nardis, Fagotti, PRL (2016)

Conserved local charges $\boldsymbol{Q} = (Q_1, Q_2, \ldots)$ with $Q_j = \int \mathrm{d}x\, q_j(x)$ and conjugate thermodynamic fields $\boldsymbol{\beta}(x,t) = (\beta_1(x,t), \beta_2(x,t), \ldots)$

Euler-scale approximation

$$\frac{\mathrm{Tr}\left[\mathrm{e}^{-\sum_j \int \mathrm{d}x'\, \beta_j(x') q_j(x')} \mathcal{O}(x,t)\right]}{\mathrm{Tr}\left[\mathrm{e}^{-\sum_j \int \mathrm{d}x'\, \beta_j(x') q_j(x')}\right]} \approx \frac{\mathrm{Tr}\left[\mathrm{e}^{-\sum_j \beta_j(x,t) Q_j} \mathcal{O}(0,0)\right]}{\mathrm{Tr}\left[\mathrm{e}^{-\sum_j \beta_j(x,t) Q_j}\right]}$$

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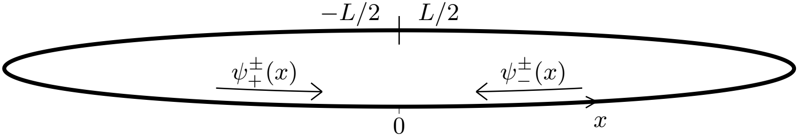
Non-local Luttinger (NLL) model

Hamiltonian: $H = \sum_{r=\pm} \int \mathrm{d}x\, v_F :\psi_r^+(x)(-\mathrm{i}r\partial_x)\psi_r^-(x):$

$$+\frac{1}{2}\sum_{r,r'=\pm}\iint \mathrm{d}x\mathrm{d}x' \left[\delta_{r,-r'}g_2V_2(x-x') + \delta_{r,r'}g_4V_4(x-x')\right]\rho_r(x)\rho_{r'}(x')$$



Fermionic fields: $\psi_r^-(x)$ and $\psi_r^+(x) = \psi_r^-(x)^\dagger$ satisfying CAR

$$\{\psi_r^-(x), \psi_{r'}^+(x')\} = \delta_{r,r'}\delta(x-x') \qquad \{\psi_r^\pm(x), \psi_{r'}^\pm(x')\} = 0$$

Densities: $\rho_\pm(x) = :\psi_\pm^+(x)\psi_\pm^-(x):$

Tomonaga, Prog. Theor. Phys. (1950)
Luttinger, JMP (1963)
Mattis, Lieb, JMP (1965)
Mastropietro, Wang, PRB (2015)
Langmann, Lebowitz, Mastropietro, P.M., CMP (2017)
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Exact solution by bosonization

Fourier transforms: $\begin{cases} V_{2,4}(p) = \int_{-L/2}^{L/2} \mathrm{d}x\, V_{2,4}(x)\mathrm{e}^{-\mathrm{i}px} \\ \rho_\pm(p) \;\; = \int_{-L/2}^{L/2} \mathrm{d}x\, \rho_\pm(x)\mathrm{e}^{-\mathrm{i}px} \end{cases}$ for $p \in \frac{2\pi}{L}\mathbb{Z}$

Conditions: $V_{2,4}(p) = V_{2,4}(-p)$ and

$$\big|g_2 V_2(p)\big| < 2\pi v_F + g_4 V_4(p) \quad \forall\, p, \qquad \sum_{p>0} \frac{p\big[g_2 V_2(p)\big]^2}{2\pi v_F\big[2\pi v_F + g_4 V_4(p)\big]} < \infty$$

Examples: $V_{2,4}(p) = \dfrac{\pi v_F}{1+(ap)^2}$ $\quad V_{2,4}(p) = \pi v_F \operatorname{sech}(ap)$ for $a > 0$

Bosonization $\implies$ H written as a bilinear in $\rho_\pm(p) = \rho_\pm(-p)^\dagger$ satisfying

$$\rho_r(p)|\Psi_0\rangle = 0 \quad \forall\, rp \geq 0, \qquad [\rho_r(p), \rho_{r'}(-p')] = r\delta_{r,r'}\frac{Lp}{2\pi}\delta_{p,p'}$$

Reviews: Voit (1995); Schulz, Cuniberti, Pieri (2000); Langmann, P.M., JMP (2015)

Diagonalization

Propagation velocity and Luttinger parameter

$$v(p) = v_F \sqrt{\left[1 + \frac{g_4 V_4(p)}{2\pi v_F}\right]^2 - \left[\frac{g_2 V_2(p)}{2\pi v_F}\right]^2}, \quad K(p) = \sqrt{\frac{2\pi v_F + g_4 V_4(p) - g_2 V_2(p)}{2\pi v_F + g_4 V_4(p) + g_2 V_2(p)}}$$

then

$$H = \sum_{r,r'=\pm} \sum_{p} \frac{\pi}{L} v(p) \frac{1 + rr' K(p)^2}{2K(p)} :\rho_r(-p)\rho_{r'}(p): - \sum_{p>0} \left[v_F - v(p)\frac{1 + K(p)^2}{2K(p)}\right] p$$

Bogoliubov transformation

$$\widetilde{\rho}_r(p) = \sum_{r'=\pm} \frac{1 + rr' K(p)}{2\sqrt{K(p)}} \rho_{r'}(p) \qquad \widetilde{Q}_r = \widetilde{\rho}_r(p = 0)$$

then

$$H = \sum_{r=\pm} \frac{\pi}{L} v(0) \widetilde{Q}_r^2 + \sum_{r=\pm} \sum_{p \neq 0} \frac{\pi}{L} v(p) :\widetilde{\rho}_r(-p)\widetilde{\rho}_r(p): + E_{\mathsf{GS}}$$

Densities and currents

Energy density: $\mathcal{E}(x) = \pi v_F \left[:\rho_+(x)^2: + :\rho_-(x)^2: \right]$

$$+\frac{1}{2} \sum_{r,r'=\pm} \int_{-L/2}^{L/2} \mathrm{d}x' \left[\delta_{r,-r'} g_2 V_2(x-x') + \delta_{r,r'} g_4 V_4(x-x') \right] \rho_r(x) \rho_{r'}(x')$$

Particle density: $\rho(x) = \rho_+(x) + \rho_-(x)$

Associated heat and charge currents: $\mathcal{J}(x)$ and $j(x)$ satisfying

$$\partial_t \mathcal{E} + \partial_x \mathcal{J} = 0 \qquad \partial_t \rho + \partial_x j = 0$$

Outline

Chemical-potential profile

Initial state $\hat{\rho}_{\mu(\cdot)} = \frac{\mathrm{e}^{-G}}{\mathrm{Tr}[\mathrm{e}^{-G}]}$ with $G = \beta\Big[H - \int_{-L/2}^{L/2} \mathrm{d}x\, \mu(x)\rho(x)\Big]$
and $\mu(x) = \mu + \delta\mu W(x)$ for smooth $W(x)$

Large and small gauge transformations implemented by

$$\mathcal{V} = \mathcal{V}_+\mathcal{V}_- \qquad \mathcal{V}_r = R_r^{-rw}\mathrm{e}^{-r\mathrm{i}\int_{-L/2}^{L/2}\mathrm{d}x\,\eta(x)\rho_r(x)}$$

If $w = \frac{L}{2\pi}\frac{K(0)\mu}{v(0)}$ and $\mathrm{i}\eta(p) = \frac{K(p)\delta\mu W(p)}{v(p)}$, then $\mathcal{V}G\mathcal{V}^{-1} = \beta H + \text{const}$

$$\Longrightarrow \boxed{\mathrm{Tr}\big[\hat{\rho}_{\mu(\cdot)}\mathcal{O}(x,t)\big] = \frac{\mathrm{Tr}[\mathrm{e}^{-\beta H}\mathcal{O}_{\mathcal{V}}(x,t)]}{\mathrm{Tr}[\mathrm{e}^{-\beta H}]}}$$

with $\mathcal{O}_{\mathcal{V}}(x,t) = \mathcal{V}\mathcal{O}(x,t)\mathcal{V}^{-1}$ for $\mathcal{O}(x,t) = \mathrm{e}^{\mathrm{i}Ht}\mathcal{O}(x)\mathrm{e}^{-\mathrm{i}Ht}$

Langmann, Lebowitz, Mastropietro, P.M., CMP (2017)

Exact results for charge transport

Particle density

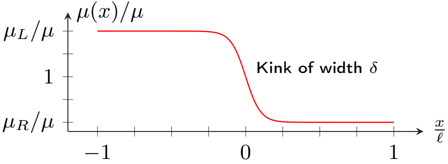
$$\lim_{L\to\infty} \mathrm{Tr}\big[\hat{\rho}_{\mu(\cdot)}\rho(x,t)\big] = \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi}\,\frac{K(p)\mu(p)}{2\pi v(p)}\big[\mathrm{e}^{\mathrm{i}p[x-v(p)t]} + \mathrm{e}^{\mathrm{i}p[x+v(p)t]}\big]$$

Charge current

$$\lim_{L\to\infty} \mathrm{Tr}\big[\hat{\rho}_{\mu(\cdot)}j(x,t)\big] = \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi}\,\frac{K(p)\mu(p)}{2\pi}\big[\mathrm{e}^{\mathrm{i}p[x-v(p)t]} - \mathrm{e}^{\mathrm{i}p[x+v(p)t]}\big]$$

Charge transport in the NLL model

Initial chemical-potential profile



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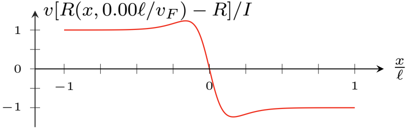

$$R(x,t) = \lim_{L\to\infty} \mathrm{Tr}\big[\hat{\rho}_{\mu(\cdot)}\rho(x,t)\big]$$

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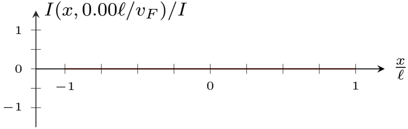

$$I(x,t) = \lim_{L\to\infty} \mathrm{Tr}\big[\hat{\rho}_{\mu(\cdot)}j(x,t)\big]$$

$$I = \lim_{t\to\infty} I(x,t)$$

Here: $V_{2,4}(p) = \pi v_F \operatorname{sech}(ap)$, $g_{2,4} = 1$, $a/\ell = 0.25$, $\mu_L = 1.95$, $\mu_R = 0.05$, $\delta/\ell = 0.1$, $v_F/\ell = 0.025$.

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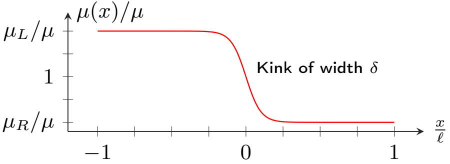

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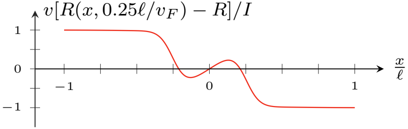

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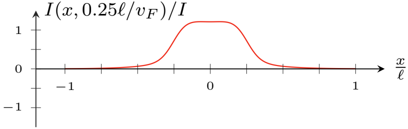

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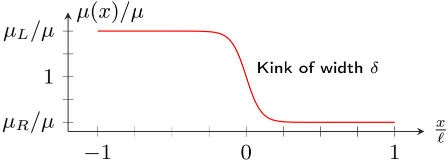

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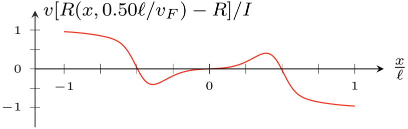

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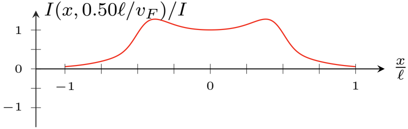

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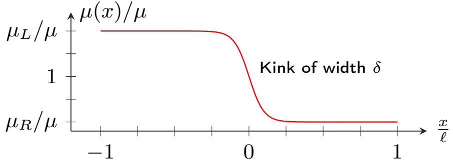

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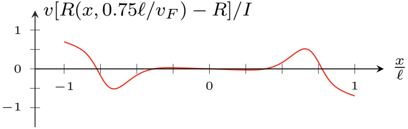

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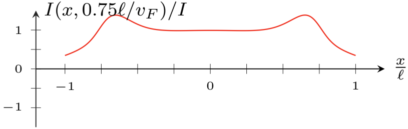

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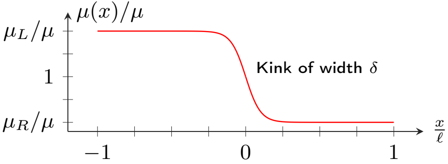

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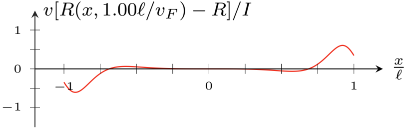

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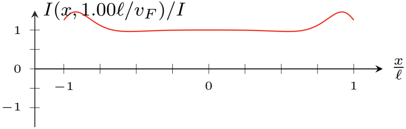

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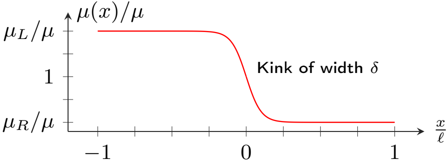

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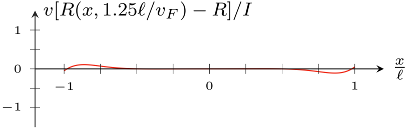

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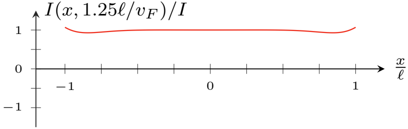

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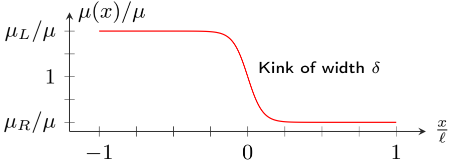

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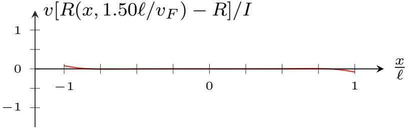

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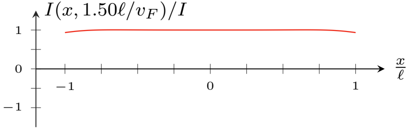

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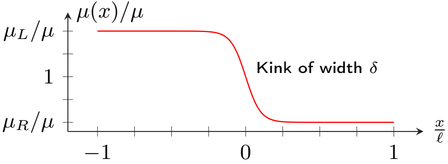

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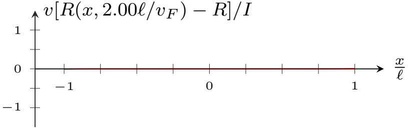

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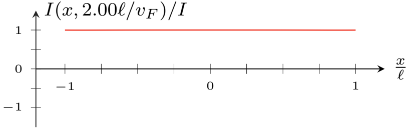

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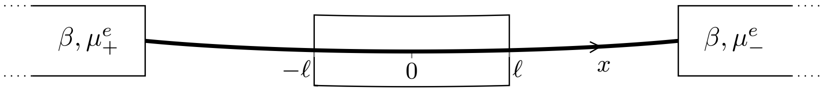
Non-equilibrium steady state (NESS)

Evolving $\hat{\rho}_{\mu(\cdot)}$ under H yields $\hat{\rho}_{\mathsf{NESS}} = \dfrac{\mathrm{e}^{-G_{\mathsf{NESS}}}}{\mathrm{Tr}\,\mathrm{e}^{-G_{\mathsf{NESS}}}}$ at long times with

$$G_{\mathsf{NESS}} = \beta\left(H - \mu_L\sqrt{K(0)}\widetilde{Q}_+ - \mu_R\sqrt{K(0)}\widetilde{Q}_-\right) = \beta\left(H - \mu_+^e Q_+ - \mu_-^e Q_-\right)$$

where $Q_\pm = \rho_\pm(p)|_{p=0}$ and $\mu_\pm^e = \dfrac{\mu_L+\mu_R}{2} \pm K(0)\dfrac{\mu_L-\mu_R}{2}$.

Steady current: $\quad I = \lim\limits_{L\to\infty} \mathrm{Tr}\big[\hat{\rho}_{\mathsf{NESS}}\, j(x)\big] = \dfrac{e^2}{2\pi\hbar}(\mu_+^e - \mu_-^e)$



Recall that $\widetilde{Q}_r = \widetilde{\rho}_r(p)|_{p=0}$ satisfy $\widetilde{Q}_r = \dfrac{1+K(0)}{2\sqrt{K(0)}}Q_r + \dfrac{1-K(0)}{2\sqrt{K(0)}}Q_{-r}$.

Inverse-temperature profile

Initial state $\hat{\rho}_{\beta(\cdot)} = \frac{\mathrm{e}^{-G}}{\mathrm{Tr}[\mathrm{e}^{-G}]}$ with $G = \int_{-L/2}^{L/2} \mathrm{d}x\, \beta(x)\mathcal{E}(x)$ and

$\beta(x) = \beta[1 + \epsilon W(x)]$ for smooth $W(x)$ and $\epsilon = \delta\beta/\beta$

Quasi-free bosonic model $\implies$ Use boson second quantization

$$\mathcal{A} = \mathrm{d}\hat{\Gamma}(A) = \sum_{r,r'} \sum_{p,p'} A_{r,r'}(p,p') :\mathfrak{b}_r^+(p)\mathfrak{b}_r^-(p):$$

of 1-particle operator $A = \big(A_{r,r'}(p,p')\big)$ with boson operators

$$\mathfrak{b}_r^-(p) = \sqrt{2\pi/L|p|}\rho_r(p) \qquad \mathfrak{b}_r^-(p)|\Psi_0\rangle = 0 \ \ \forall\, rp \geq 0$$

$$\mathfrak{b}_r^+(p) = r\,\mathrm{sgn}(p)\mathfrak{b}_r^-(p)^\dagger \qquad \big[\mathfrak{b}_r^-(p), \mathfrak{b}_{r'}^+(p')\big] = \delta_{r,r'}\delta_{p,p'}$$

1-particle trace: $\mathrm{tr}(A) = \sum_r \sum_p A_{r,r}(p,p)$

Langmann, Lebowitz, Mastropietro, P.M., PRB (2017)

Formal series expansion

Let $G = \beta(H + \mathcal{W})$, $H = \mathrm{d}\hat{\Gamma}(K)$, $\mathcal{W} = \mathrm{d}\hat{\Gamma}(W)$, $\mathcal{O} = \mathrm{d}\hat{\Gamma}(O)$. Then

$$
\begin{aligned}
\frac{\mathrm{Tr}\left[\mathrm{e}^{-G}\mathcal{O}\right]}{\mathrm{Tr}\left[\mathrm{e}^{-G}\right]} - \frac{\mathrm{Tr}\left[\mathrm{e}^{-\beta H}\mathcal{O}\right]}{\mathrm{Tr}\left[\mathrm{e}^{-\beta H}\right]} &= \frac{\mathrm{Tr}\left[\mathrm{e}^{-\beta \mathrm{d}\hat{\Gamma}(K+W)}\mathrm{d}\hat{\Gamma}(O)\right]}{\mathrm{Tr}\left[\mathrm{e}^{-\beta \mathrm{d}\hat{\Gamma}(K+W)}\right]} - \frac{\mathrm{Tr}\left[\mathrm{e}^{-\beta \mathrm{d}\hat{\Gamma}(K)}\mathrm{d}\hat{\Gamma}(O)\right]}{\mathrm{Tr}\left[\mathrm{e}^{-\beta \mathrm{d}\hat{\Gamma}(K)}\right]} \\
&= \mathrm{tr}\left(\left\{\left[\mathrm{e}^{2\beta(K+W)} - 1\right]^{-1} - \left[\mathrm{e}^{2\beta K} - 1\right]^{-1}\right\} O\right) \\
&= \frac{1}{2}\,\mathrm{tr}\left(\left\{\coth(\beta[K+W]) - \coth(\beta K)\right\} O\right) \\
&= \frac{1}{\beta} \sum_{\nu \in (2\pi/\beta)\mathbb{Z}} \mathrm{tr}\left(\left\{\left[\mathrm{i}\nu - 2(K+W)\right]^{-1} - \left[\mathrm{i}\nu - 2K\right]^{-1}\right\} O\right) \\
&= \sum_{n=1}^{\infty} \frac{1}{\beta} \sum_{\nu \in (2\pi/\beta)\mathbb{Z}} \mathrm{tr}\left(\left[\mathrm{i}\nu - 2K\right]^{-1}\left\{2W\left[\mathrm{i}\nu - 2K\right]^{-1}\right\}^{n} O\right)
\end{aligned}
$$

$$
\Longrightarrow \quad \boxed{\frac{\mathrm{Tr}\left[\mathrm{e}^{-G}\mathcal{O}(x,t)\right]}{\mathrm{Tr}\left[\mathrm{e}^{-G}\right]} = \frac{\mathrm{Tr}\left[\mathrm{e}^{-\beta H}\mathcal{O}(x,t)\right]}{\mathrm{Tr}\left[\mathrm{e}^{-\beta H}\right]} + \sum_{n=1}^{\infty} \epsilon^{n} G_{\mathcal{O}}^{(n)}(x,t)}
$$

Langmann, Lebowitz, Mastropietro, P.M., PRB (2017)

Both inverse-temperature and chemical-potential profiles

Initial state $\hat{\rho}_{\beta(\cdot),\mu(\cdot)} = \frac{\mathrm{e}^{-G}}{\mathrm{Tr}[\mathrm{e}^{-G}]}$ with $G = \int_{-L/2}^{L/2} \mathrm{d}x\, \beta(x) \big[\mathcal{E}(x) - \mu(x)\rho(x) \big]$

Conjugate $\mathcal{O} = \mathrm{d}\hat{\Gamma}(O)$ with $\mathcal{V}$ implementing gauge transformations

$$\mathcal{O} \to \mathcal{O}_{\mathcal{V}} = \mathcal{V}\mathcal{O}\mathcal{V}^{-1} = \mathrm{d}\hat{\Gamma}(O_V) + \text{1st and 0th order terms in } \mathfrak{b}_r^-(p)$$

with $O_V = VOV^{-1}$ and $\mathcal{V}$ depending on $\beta(x)$ and $\mu(x)$

Repeat formal series expansion

$$\Longrightarrow \boxed{\frac{\mathrm{Tr}\big[\mathrm{e}^{-G}\mathcal{O}(x,t)\big]}{\mathrm{Tr}\big[\mathrm{e}^{-G}\big]} = \frac{\mathrm{Tr}\big[\mathrm{e}^{-\beta H}\mathcal{O}_{\mathcal{V}}(x,t)\big]}{\mathrm{Tr}\big[\mathrm{e}^{-\beta H}\big]} + \sum_{n=1}^{\infty} \epsilon^n G_{\mathcal{O}_{\mathcal{V}}}^{(n)}(x,t)}$$

Local Luttinger model

The terms for $\mathcal{O} = \mathcal{E}$ and $\mathcal{J}$ can be evaluated and resummed:

$$\lim_{L\to\infty} \mathrm{Tr}\big[\hat\rho_{\beta(\cdot),\mu(\cdot)}\mathcal{E}(x,t)\big] = \frac{F(x-vt)+F(x+vt)}{2v}$$

$$\lim_{L\to\infty} \mathrm{Tr}\big[\hat\rho_{\beta(\cdot),\mu(\cdot)}\mathcal{J}(x,t)\big] = \frac{F(x-vt)-F(x+vt)}{2}$$

where $F(x) = \frac{\pi}{6\beta^2} + \frac{K\mu(x)^2}{2\pi} + \sum_{n=1}^{\infty} \epsilon^n F_n(x)$ and

$$F_n(x) = \int_{\mathbb{R}^{n+1}} \frac{dp_0 \ldots dp_n}{(2\pi)^{n+1}} \left(\prod_{j=0}^{n-1} \hat W(p_j - p_{j+1})\right)$$

$$\times \frac{v}{2}\sum_r \frac{1}{\beta} \sum_{\nu\in(2\pi/\beta)\mathbb{Z}} \left(\prod_{j=0}^{n} \frac{rvp_j}{i\nu - rvp_j}\right) e^{i(p_0-p_n)x}$$

implies $$F(x) = \frac{\pi c}{6\beta(x)^2} + \frac{K\mu(x)^2}{2\pi} + \frac{cv^2}{12\pi}\left[\frac{\beta''(x)}{\beta(x)} - \frac{1}{2}\left(\frac{\beta'(x)}{\beta(x)}\right)^2\right]$$

with $c = 1$, $v = v(0)$, $K = K(0)$. Langmann, Lebowitz, Mastropietro, P.M., PRB (2017)
Gawedzki, Langmann, P.M., JSP (2018)

Local Luttinger model

The terms for $\mathcal{O} = \mathcal{E}$ and $\mathcal{J}$ can be evaluated and resummed:

$$\lim_{L\to\infty} \mathrm{Tr}\big[\hat{\rho}_{\beta(\cdot),\mu(\cdot)}\mathcal{E}(x,t)\big] = \frac{F(x-vt)+F(x+vt)}{2v}$$
$$\lim_{L\to\infty} \mathrm{Tr}\big[\hat{\rho}_{\beta(\cdot),\mu(\cdot)}\mathcal{J}(x,t)\big] = \frac{F(x-vt)-F(x+vt)}{2}$$

while all terms except the first are zero for $\mathcal{O} = \rho$ and j:

$$\lim_{L\to\infty} \mathrm{Tr}\big[\hat{\rho}_{\beta(\cdot),\mu(\cdot)}\rho(x,t)\big] = \frac{G(x-vt)+G(x+vt)}{2v}$$
$$\lim_{L\to\infty} \mathrm{Tr}\big[\hat{\rho}_{\beta(\cdot),\mu(\cdot)}j(x,t)\big] = \frac{G(x-vt)-G(x+vt)}{2}$$

where
$$F(x) = \frac{\pi c}{6\beta(x)^2} + \frac{K\mu(x)^2}{2\pi} + \frac{cv^2}{12\pi}\left[\frac{\beta''(x)}{\beta(x)} - \frac{1}{2}\left(\frac{\beta'(x)}{\beta(x)}\right)^2\right]$$
$$G(x) = \frac{K}{\pi}\mu(x)$$

with $c = 1$, $v = v(0)$, $K = K(0)$.

Langmann, Lebowitz, Mastropietro, P.M., PRB (2017)
Gawedzki, Langmann, P.M., JSP (2018)

Conformal field theory

The term $-\frac{\beta''(x)}{\beta(x)} + \frac{1}{2}\left(\frac{\beta'(x)}{\beta(x)}\right)^2$ can be written as a Schwarzian derivative

$$(Sg)(x) = \frac{g'''(x)}{g'(x)} - \frac{3}{2}\left(\frac{g''(x)}{g'(x)}\right)^2$$

of the orientation-preserving circle diffeomorphism

$$g(x) = \int_0^x \mathrm{d}x' \, \frac{\beta_0}{\beta(x')}, \quad \frac{1}{\beta_0} = \frac{1}{L}\int_{-L/2}^{L/2} \frac{\mathrm{d}x}{\beta(x)}.$$

Suggests that the results for $\beta(x)$ can be obtained by conformal transformations: Shown using projective unitary representations of $g(x)$ on the Hilbert space. Gawedzki, Langmann, P.M., JSP (2018)

For inhomogeneous systems with velocity $v(x)$. P.M., accepted in AHP (2021)

Outline

Conserved charges

Let $\mathcal{I}_1 = \{(r,p) \,|\, r = \pm,\ p \in \frac{2\pi}{L}\mathbb{Z}^+\}$ and $\mathcal{I}_0 = \{(r,0) \,|\, r = \pm\}$

Mutually commuting conserved charges

$$Q_{r',p'} = q_{r',p'}(0), \quad q_{r',p'}(p) = \begin{cases} \frac{\pi}{L} v(p') \big[:\widetilde{\rho}_{r'}(p-p')\widetilde{\rho}_{r'}(p'): \\ \qquad + :\widetilde{\rho}_{r'}(-p')\widetilde{\rho}_{r'}(p+p'): \big] & (r',p') \in \mathcal{I}_1 \\ \frac{\pi}{L} v(0)\widetilde{\rho}_{r'}(p)\widetilde{Q}_{r'} & (r',p') \in \mathcal{I}_0 \end{cases}$$

$$Q^J_{r'} = q^J_{r'}(0), \qquad q^J_{r'}(p) = \sqrt{K(p)}\,\widetilde{\rho}_{r'}(p) \qquad r' = \pm$$

Generalized Gibbs ensemble

$$\boldsymbol{Q} = \Big((Q_{r',p'})_{(r',p')\in\mathcal{I}}, (Q^J_{r'})_{r'=\pm} \Big) \qquad \mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_0$$

such that $H = \sum_{(r',p')\in\mathcal{I}} Q_{r',p'} + E_{\mathsf{GS}}$

Inhomogeneous quantum quench

Initial state $\dfrac{\mathrm{e}^{-G_{\boldsymbol{\beta}(\cdot)}}}{\mathrm{Tr}\big[\mathrm{e}^{-G_{\boldsymbol{\beta}(\cdot)}}\big]}$ with

$$G_{\boldsymbol{\beta}(\cdot)} = \sum_{(r',p')\in\mathcal{I}_1} \int_{-L/2}^{L/2} \mathrm{d}x \, \beta_{r',p'}(x) q_{r',p'}(x) + \sum_{r'=\pm} \int_{-L/2}^{L/2} \mathrm{d}x \, \beta_{r',0}(x) [q_{r',0}(x) - \mu^J_{r'}(x) q^J_{r'}(x)]$$

where $q_{r',p'}(x) = \sum_p \frac{1}{L} q_{r',p'}(p) \mathrm{e}^{\mathrm{i}px}$ and similarly for $q^J_{r'}(x)$

Quench dynamics:

$$\langle \mathcal{O}(x,t) \rangle_{\boldsymbol{\beta}(\cdot)} = \frac{\mathrm{Tr}\big[\mathrm{e}^{-G_{\boldsymbol{\beta}(\cdot)}} \mathcal{O}(x,t)\big]}{\mathrm{Tr}\big[\mathrm{e}^{-G_{\boldsymbol{\beta}(\cdot)}}\big]} \qquad \mathcal{O}(x,t) = \mathrm{e}^{\mathrm{i}Ht} \mathcal{O}(x) \mathrm{e}^{-\mathrm{i}Ht}$$

Euler-scale GHD

Euler-scale hydrodynamic approximation

$$\langle \mathcal{O}(x,t)\rangle_{\boldsymbol{\beta}(\cdot)} \approx \langle \mathcal{O}\rangle_{\boldsymbol{\beta}(x,t)} = \frac{\mathrm{Tr}\left[\mathrm{e}^{-G_{\boldsymbol{\beta}(x,t)}}\mathcal{O}\right]}{\mathrm{Tr}\left[\mathrm{e}^{-G_{\boldsymbol{\beta}(x,t)}}\right]} \qquad \mathcal{O} = \mathcal{O}(0,0)$$

with

$$G_{\boldsymbol{\beta}(x,t)} = \sum_{(r',p')\in\mathcal{I}\setminus\mathcal{I}_0} \beta_{r',p'}(x,t) Q_{r',p'} + \sum_{r'=\pm} \beta_{r',0}(x,t)[Q_{r',0} - \mu^J_{r'}(x,t) Q^J_{r'}]$$

Hydrodynamic equations $\partial_t \langle q_i\rangle_{\boldsymbol{\beta}(x,t)} + \partial_x \langle j_i\rangle_{\boldsymbol{\beta}(x,t)} = 0$ become

$$\partial_t \langle q_{r',p'}\rangle_{\boldsymbol{\beta}(x,t)} + \sum_{(r'',p'')\in\mathcal{I}} A_{r',p'}^{r'',p''}(x,t)\,\partial_x \langle q_{r'',p''}\rangle_{\boldsymbol{\beta}(x,t)} = 0$$

$$\partial_t \langle q^J_{r'}\rangle_{\boldsymbol{\beta}(x,t)} + \sum_{r''=\pm} A_{r'}^{r''}(x,t)\,\partial_x \langle q^J_{r''}\rangle_{\boldsymbol{\beta}(x,t)} = 0$$

P.M., SciPost Phys. (2020)

Explicit solutions

Flux Jacobians:

$$A_{r',p'}^{r'',p''}(x,t) = \frac{\partial \langle j_{r',p'} \rangle_{\boldsymbol{\beta}(x,t)}}{\partial \langle q_{r'',p''} \rangle_{\boldsymbol{\beta}(x,t)}} = v^{\mathsf{eff}}_{r',p'} \delta_{r',r''} \delta_{p',p''}$$

$$A_{r'}^{r''}(x,t) = \frac{\partial \langle j^J_{r'} \rangle_{\boldsymbol{\beta}(x,t)}}{\partial \langle q^J_{r''} \rangle_{\boldsymbol{\beta}(x,t)}} = v^{\mathsf{eff}}_{r',0} \delta_{r',r''}$$

$$v^{\mathsf{eff}}_{r,p} = r \frac{\mathrm{d}[v(p)p]}{\mathrm{d}p}$$

Hydrodynamic equations $\implies$ PDEs for $\boldsymbol{\beta}(x,t)$ with $\boldsymbol{\beta}(x,0) = \boldsymbol{\beta}(x)$:

$$\partial_t \langle q_{r',p'} \rangle_{\boldsymbol{\beta}(x,t)} + v^{\mathsf{eff}}_{r',p'} \partial_x \langle q_{r',p'} \rangle_{\boldsymbol{\beta}(x,t)} = 0$$

$$\partial_t \langle q^J_{r'} \rangle_{\boldsymbol{\beta}(x,t)} + v^{\mathsf{eff}}_{r',0} \partial_x \langle q^J_{r'} \rangle_{\boldsymbol{\beta}(x,t)} = 0$$

Solutions:

$$\boxed{\beta_{r',p'}(x,t) = \beta_{r',p'}(x - v^{\mathsf{eff}}_{r',p'} t) \qquad \mu^J_{r'}(x,t) = \mu^J_{r'}(x - v^{\mathsf{eff}}_{r',0} t)}$$

P.M., SciPost Phys. (2020)

Outline

- Closed 1+1D systems out of equilibrium
- Non-local Luttinger model
- Exact analytical transport results
- Generalized hydrodynamics
- Emergence of hydrodynamics

Euler-scale GHD results for transport

For simplicity, initial state with $\beta_{\pm,p}(x) = \beta(x)$ and $\mu^J_\pm(x) = \mu(x)$

Heat transport:

$$\langle \mathcal{E} \rangle^\infty_{\boldsymbol{\beta}(x,t)} = \sum_r \int_0^\infty \frac{\mathrm{d}p}{2\pi} \frac{v(p)p}{\mathrm{e}^{\beta(x-v^{\mathsf{eff}}_{r,p}t)v(p)p} - 1} + \sum_r \frac{K(0)\mu(x - v^{\mathsf{eff}}_{r,0}t)^2}{4\pi v(0)} + \mathcal{E}_{\mathsf{GS}}$$

$$\langle \mathcal{J} \rangle^\infty_{\boldsymbol{\beta}(x,t)} = \sum_r \int_0^\infty \frac{\mathrm{d}p}{2\pi} v^{\mathsf{eff}}_{r,p} \frac{v(p)p}{\mathrm{e}^{\beta(x-v^{\mathsf{eff}}_{r,p}t)v(p)p} - 1} + \sum_r \frac{rK(0)\mu(x - v^{\mathsf{eff}}_{r,0}t)^2}{4\pi}$$

Charge transport:

$$\langle \rho \rangle^\infty_{\boldsymbol{\beta}(x,t)} = \sum_r \frac{K(0)\mu(x - v^{\mathsf{eff}}_{r,0}t)}{2\pi v(0)}$$

$$\langle j \rangle^\infty_{\boldsymbol{\beta}(x,t)} = \sum_r \frac{rK(0)\mu(x - v^{\mathsf{eff}}_{r,0}t)}{2\pi}$$

Here: $\langle \cdot \rangle^\infty = \lim_{L\to\infty} \langle \cdot \rangle$

Emergent Euler-scale GHD results

Claim:

$$\lim_{\lambda\to\infty}\langle\mathcal{O}(\lambda x,\lambda t)\rangle_{\boldsymbol{\beta}(\cdot/\lambda)} = \langle\mathcal{O}\rangle_{\boldsymbol{\beta}(x,t)}$$

From exact results for charge transport:

$$\lim_{\lambda\to\infty}\langle\rho(\lambda x,\lambda t)\rangle^{\infty}_{\boldsymbol{\beta}(\cdot/\lambda)} = \lim_{\lambda\to\infty}\sum_r\int_{-\infty}^{\infty}\frac{\mathrm{d}p}{2\pi}\frac{K(p)\lambda\mu(\lambda p)}{2\pi v(p)}\mathrm{e}^{\mathrm{i}p[\lambda x - rv(p)\lambda t]} + O(\epsilon^2)$$

$$= \sum_r \frac{K(0)\mu(x - v^{\mathsf{eff}}_{r,0}t)}{2\pi v(0)} + O(\epsilon^2) = \langle\rho\rangle^{\infty}_{\boldsymbol{\beta}(x,t)} + O(\epsilon^2)$$

$$\lim_{\lambda\to\infty}\langle j(\lambda x,\lambda t)\rangle^{\infty}_{\boldsymbol{\beta}(\cdot/\lambda)} = \lim_{\lambda\to\infty}\sum_r\int_{-\infty}^{\infty}\frac{\mathrm{d}p}{2\pi}\frac{rK(p)\lambda\mu(\lambda p)}{2\pi}\mathrm{e}^{\mathrm{i}p[\lambda x - rv(p)\lambda t]} + O(\epsilon^2)$$

$$= \sum_r \frac{rK(0)\mu(x - v^{\mathsf{eff}}_{r,0}t)}{2\pi} + O(\epsilon^2) = \langle j\rangle^{\infty}_{\boldsymbol{\beta}(x,t)} + O(\epsilon^2)$$

Similar for heat transport

Also: Formal proof for any local observable (no series expansion in ϵ)

Euler scale vs. long-range interactions

On the previous slide, change of variables from p to p/λ

$\implies$ Dependence on $K(p/\lambda)$ and $v(p/\lambda)$

$\implies$ Dependence on $V_{2,4}(p/\lambda)$

Example: If $V_{2,4}(p) = \frac{\pi v_F}{1+(ap)^2}$ then

$$V_{2,4}(p/\lambda) \to \begin{cases} 0 & a \to \infty \text{ (for } \lambda \text{ fixed)} \\ \pi v_F & \lambda \to \infty \text{ (for } a \text{ fixed)} \end{cases}$$

in the sense of distributions

$\implies$ Euler-scale and long-range limits do not commute

Summary

- Exact analytical results for heat and charge transport in the non-local Luttinger model. Non-trivial dispersive effects.
- Emergence of Euler-scale generalized hydrodynamics in the non-local Luttinger model with short-range interactions.

Outlook:

- General quasi-free bosonic models in 1+1 dimensions. Effects of integrability-breaking terms?
- Euler scale vs. long-range interactions. Precise conditions?
- Beyond Euler-scale hydrodynamics.
- Global quenches, e.g., interaction quench.